

WELL-POSEDNESS FOR MULTICOMPONENT SCHRÖDINGER–gKdV SYSTEMS AND STABILITY OF SOLITARY WAVES WITH PRESCRIBED MASS

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ABSTRACT. In this paper we prove the well-posedness issues of the associated initial value problem, the existence of nontrivial solutions with prescribed L^2 -norm, and the stability of associated solitary waves for two classes of coupled nonlinear dispersive equations. The first problem here describes the nonlinear interaction between two Schrödinger type short waves and a generalized Korteweg-de Vries type long wave and the second problem describes the nonlinear interaction of two generalized Korteweg-de Vries type long waves with a common Schrödinger type short wave. The results here extend many of the previously obtained results for two-component coupled Schrödinger-Korteweg-de Vries systems.

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Mathematics Subject Classification. 35Q53, 35Q55, 35B35, 35B65, 35A15.

Keywords. Schrödinger-KdV equations, local and global well-posedness, smoothing effects, Bourgain space, normalized solutions, solitary waves, stability, variational methods.

1. Introduction

In this paper, consideration is given to multicomponent nonlinear systems describing the interaction between long and short dispersive waves. First we are concerned with a 3-wave system describing the interaction of two nonlinear Schrödinger (NLS)-type short waves with a generalized Korteweg-de Vries (gKdV)-type long wave and the second system we study contains two gKdV-type long wave fields and a common NLS-type short wave. The first problem considered here has the form

$$\begin{cases} i\partial_t u_1 + \partial_x^2 u_1 + \gamma_1 |u_1|^{q_1} u_1 = -\alpha_1 u_1 v, \\ i\partial_t u_2 + \partial_x^2 u_2 + \gamma_2 |u_2|^{q_2} u_2 = -\alpha_2 u_2 v, \\ \partial_t v + \partial_x^3 v + \beta v^p \partial_x v = -\frac{1}{2} \partial_x (\alpha_1 |u_1|^2 + \alpha_2 |u_2|^2), \end{cases} \quad (1.1)$$

and the second system generally takes the form

$$\begin{cases} i\partial_t u + \partial_x^2 u + \gamma |u|^q u = -\alpha_1 u v_1 - \alpha_2 u v_2, \\ \partial_t v_1 + \partial_x^3 v_1 + \beta_1 v_1^{p_1} \partial_x v_1 = -\frac{1}{2} \alpha_1 \partial_x (|u|^2), \\ \partial_t v_2 + \partial_x^3 v_2 + \beta_2 v_2^{p_2} \partial_x v_2 = -\frac{1}{2} \alpha_2 \partial_x (|u|^2), \end{cases} \quad (1.2)$$

where u, u_1 , and u_2 are \mathbb{C} -valued functions of $(x, t) \in \mathbb{R}^2$; v, v_1 , and v_2 are \mathbb{R} -valued functions of $(x, t) \in \mathbb{R}^2$; and the constants $\alpha_j, \gamma_j, \gamma, \beta_j$ and β are reals which depend on the context in which the system of equations have been derived. Here v, v_1, v_2 characterize long-wave fields and u, u_1, u_2 represent short wave envelopes. This type of phenomenon has been predicted in a variety of contexts in fluid mechanics, plasma physics, nonlinear optics, acoustics, to mention but a few (for an excellent list of references, the reader may consult [1, 7]). Throughout this paper we refer to the systems (1.1) and (1.2) simply as $(2+1)$ -component NLS-gKdV and $(1+2)$ -component NLS-gKdV systems, respectively.

System (1.1) admits three conserved quantities, i.e., time independent quantities, which will play an important role in this paper. The first conserved quantity for (1.1) is the energy functional E defined by

$$E(\Delta) = \int_{-\infty}^{\infty} \left(\sum_{j=1}^2 (|\partial_x u_j|^2 - \tau_j |u_j|^{q_j+2} - \alpha_j |u_j|^2 v) + |\partial_x v|^2 - \tau v^{p+2} \right) dx \quad (1.3)$$

where $\Delta = (u_1, u_2, v)$, and $\tau_j, 1 \leq j \leq 2$, and τ are given by

$$\tau_j = \frac{2\gamma_j}{q_j + 2} \quad \text{and} \quad \tau = \frac{2\beta}{(p+1)(p+2)}.$$

Other two conserved quantities for the flow defined by (1.1) are

$$H(\Delta) = \int_{-\infty}^{\infty} v^2 dx + 2 \operatorname{Im} \int_{-\infty}^{\infty} \sum_{j=1}^2 u_j \overline{\partial_x u_j} dx, \quad (1.4)$$

where the bar denotes complex conjugation and Im denotes the imaginary part of the complex function, and the component masses

$$Q(u_j) = \int_{-\infty}^{\infty} |u_j|^2 dx, \quad j = 1, 2. \quad (1.5)$$

The first conserved quantity for (1.2) is the energy functional K defined by

$$K(U) = \int_{-\infty}^{\infty} \left(\sum_{j=1}^2 \left(|\partial_x v_j|^2 - b_j v_j^{p_j+2} - \alpha_j |u|^2 v_j \right) + |\partial_x u|^2 - a |u|^{q+2} \right) dx \quad (1.6)$$

where $U = (u, v_1, v_2)$, and $b_j, 1 \leq j \leq 2$, and a are given by

$$b_j = \frac{2\beta_j}{(p_j + 1)(p_j + 2)} \quad \text{and} \quad a = \frac{2\gamma}{q + 2}.$$

Other two conservation laws of (1.2) associated with symmetries are

$$G(U) = \int_{-\infty}^{\infty} (v_1^2 + v_2^2) dx + 2 \text{Im} \int_{-\infty}^{\infty} u \overline{\partial_x u} dx, \quad (1.7)$$

which arises from the invariance of (1.2) under space translations $x \rightarrow x + \theta$, and the component mass

$$Q(u) = \int_{-\infty}^{\infty} |u|^2 dx, \quad (1.8)$$

which arises from the invariance of (1.2) under phase shifts $u \rightarrow e^{i\theta} u$.

The first purpose of this paper is to consider the question of well-posedness of the initial value problem (IVP) associated to the systems (1.1) and (1.2). We adapt the standard notion of the well-posedness in the sense of J. Hadamard, which includes existence, uniqueness, persistence property (i.e., the solution is uniquely determined and it has the same regularity as the initial data), and continuous dependence of the solution upon the given data.

The IVP associated to the $(1 + 1)$ -component NLS-KdV system has been studied extensively in the literature. In the case when $u_2 \equiv 0$, $2p = q_1 = 2$, $\beta = 1$, and $\gamma_1 \in \mathbb{R}$, the local well-posedness was studied in [29, 5]. Here the cases $\gamma_1 = 0$ and $\gamma_1 \neq 0$ describe the resonant and non-resonant interactions, respectively. In the resonant case, Guo and Miao [19] established the global well-posedness result of $(1 + 1)$ -component NLS-KdV system in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. In [28], Pecher improved these results and obtained the local well-posedness for the data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > 0$ and the global-posedness for $(u_{10}, v_0) \in H^{\frac{3}{5}+}(\mathbb{R}) \times H^{\frac{3}{5}+}(\mathbb{R})$ in the resonant case and for $(u_{10}, v_0) \in H^{\frac{2}{3}+}(\mathbb{R}) \times H^{\frac{2}{3}+}(\mathbb{R})$ in the non-resonant case. In [15], Corcho and Linares improved the local well-posedness result obtained in [28] to a larger region of the Sobolev indices. Recently, Wu [30] obtained the best local well-posedness result for the $(1 + 1)$ -component NLS-KdV system in the resonant case. Our aim here is to obtain analogous results to the full system of equations (1.1) and (1.2), considering general power nonlinearities, in the Sobolev spaces of the form $H^s \times H^s \times H^k$ and $H^s \times H^k \times H^k$, respectively.

The well-posedness issues here are addressed considering two different cases, viz., general power type and integer power type nonlinearities. For the general power type nonlinearities, we use smoothing effects of the associated linear groups combined with the maximal function type estimates to prove the local well-posedness in the energy space $H^1 \times H^1 \times H^1$. Also, with certain restriction on the indices of nonlinearity, we obtain global solution in this space. To be precise, the lack of L^2 -conserved quantity for the gKdV part requires this restriction on the power of nonlinearities (see Theorems 2.1 and 2.2 and their proofs below). In the particular case when the indices of nonlinearities $q_1 = q_2 = 2p = 2$ for the system (1.1) and $2p_1 = 2p_2 = q = 2$ for the system (1.2), we use the estimates obtained in [30] in the framework of Bourgain spaces to get local well-posedness results for the less regular data (see Theorems 2.3 and 2.4).

Next, attention will be focused to prove the existence of nontrivial (i.e., all components non-zero) solutions $(\sigma_1, \sigma_2, c, \phi_1, \phi_2, w) \in \mathbb{R}_+^2 \times \mathbb{R} \times \mathcal{H}$ of the system of equations

$$\begin{cases} -\phi_1'' + \sigma_1 \phi_1 = \gamma_1 |\phi_1|^{q_1} \phi_1 + \alpha_1 \phi_1 w, \\ -\phi_2'' + \sigma_2 \phi_2 = \gamma_2 |\phi_2|^{q_2} \phi_2 + \alpha_2 \phi_2 w, \\ -w'' + cw = \frac{\beta}{p+1} w^{p+1} + \sum_{j=1}^2 \frac{\alpha_j}{2} |\phi_j|^2. \end{cases} \quad (1.9)$$

System of ordinary differential equations (1.9) can be considered as the defining equation for travelling solitary waves of (1.1). Solitary waves of interest here have the form

$$\begin{cases} u_1(x, t) = e^{i\omega_1 t} e^{ic(x-ct)/2} \phi_1(x-ct), \\ u_2(x, t) = e^{i\omega_2 t} e^{ic(x-ct)/2} \phi_2(x-ct), \\ v(x, t) = w(x-ct), \end{cases} \quad (1.10)$$

where $\phi_1, \phi_2 : \mathbb{R} \rightarrow \mathbb{C}$, $w : \mathbb{R} \rightarrow \mathbb{R}$ all vanish at $\pm\infty$, and the parameters ω_1, ω_2, c are reals. Substituting solitary waves ansatz (1.10) into (1.1), one easily finds that (ϕ_1, ϕ_2, w) satisfies the time-independent 3-component NLS-gKdV system (1.9) with $\xi = x - ct$ and $\sigma_j = \omega_j - c^2/4$.

Given any $(r, l, m) \in \mathbb{R}_+^3$, we look for solutions (ϕ_1, ϕ_2, w) of (1.9) satisfying the condition

$$\|\phi_1\|_{L^2}^2 = r, \quad \|\phi_2\|_{L^2}^2 = l, \quad \text{and} \quad \|w\|_{L^2}^2 = m. \quad (1.11)$$

These type of solutions are of particular interest in physics. In the literature, these solutions are sometimes referred to as L^2 -normalized solutions. To infer the existence of such solutions, we study the constrained variational problem of finding, for given $(r, l, m) \in \mathbb{R}_+^3$, the extremum of the functional E over the set $S_r \times S_l \times K_m$, where for any $\lambda > 0$ we define

$$S_\lambda := \{u \in H_{\mathbb{C}}^1 : \|u\|_{L^2}^2 = \lambda\} \quad \text{and} \quad K_\lambda := \{u \in H_{\mathbb{R}}^1 : \|u\|_{L^2}^2 = \lambda\}.$$

The key ingredient in the proof the existence of minimizers is the concentration compactness lemma introduced by P.L. Lions [23]. The parameters σ_1, σ_2 , and c , in this situation, appear as Lagrange multipliers associated with the constraints.

Several work has been done in the last few years on the existence problem for solutions of coupled nonlinear systems such as (1.9). All these works have been mainly focused on (1+1)-component coupled systems such as NLS-NLS and NLS-KdV systems. Moreover, most works treat the problem in which the parameters such as σ_1, σ_2, c are being fixed. There are very few papers which deal with the existence problem of prescribed L^2 -norm solutions, for instance, see [1, 2, 4, 17, 27] for the results on prescribed L^2 -norm solutions to two-component coupled systems. Up to our knowledge, [8, 20] are the only available works which obtain prescribed L^2 -norm solutions for coupled nonlinear systems with three or more equations. The techniques in [8] follow the ideas used in [2] to obtain existence and stability results of L^2 -normalized solitary waves for three component nonlinear Schrödinger system. In [20], a different technique was used to prove the stability of the set of minimizers to a certain minimizing problem under multiconstraint conditions. In the present paper the situation is substantially different compared to that of [8, 20] due to the presence of the additional conserved quantity $H(f, g, h)$. Here we need to tackle two different variational problems in order to establish the stability result. Finally, we also mention the papers [7, 14] where different techniques were used to prove the existence of bound state solutions for multi-component NLS-KdV systems. Our final goal is to study the stability properties of solitary wave solutions of (1.1). The mathematically exact stability theory for travelling solitary waves began with a 1972 paper of T. B. Benjamin ([6]) for the KdV equation

$$\partial_t u + u \partial_x u + \partial_x^3 u = 0. \quad (1.12)$$

According to Benjamin, if $U(x, t)$ is a solution of (1.12) whose initial profile $U(x, 0) = U_0(x)$ is sufficiently close (in an appropriate function space) to a KdV solitary wave $u(x, t) = \varphi_C(x - Ct)$, where $\varphi_C(x)$ defined as

$$\varphi_C(x) = \frac{3C}{\cosh^2\left(\frac{1}{2}\sqrt{C} x\right)};$$

then the quantity

$$\inf_{x \in \mathbb{R}} \sup_{x \in \mathbb{R}} |U(x, t) - \varphi_C(x + y)| \quad (1.13)$$

will remain small for all times $t \geq 0$. Similar stability theorems have since been proved for solitary-wave solutions of many other nonlinear wave equations. Notice that the quantity (1.13) measures the difference in sup norm between the profile $u(x, t)$ for fixed t and the orbit consisting of all translates of φ_C . Since, for system (1.1), we do not know if for given phases ω_1, ω_2 and wave speed c , solitary-wave solutions are unique up to translation, we use the notion of stability in a broad sense: namely, the stability of a set consisting of possibly different solitary-wave profiles functions rather than the stability of the set of translates of a single solitary-wave profile. The precise details of our stability results are contained in Section 2 (see Theorem 2.6 and 2.8).

The structure of the paper is as follows. In Section 2, we start with some notations that will be used throughout the paper and provide the statement of main results. Section 3 addresses the issues of well-posedness theory. In Section 4, we prove the

existence result for L^2 normalized solitary-wave solutions for $(2+1)$ -component NLS-gKdV system. Finally, Section 5 studies an alternative variational characterization of solitary waves, along with their stability properties.

2. Statement of Main Results

In this section, we introduce some notations and function spaces that will be used throughout the paper and state our main results.

2.1. Notations and assumptions. We denote by \mathbb{R}_+ the set $\{x \in \mathbb{R} : x > 0\}$ and by S^1 the set $\{z \in \mathbb{C} : z = e^{i\theta}, \theta \in \mathbb{R}\}$. For $1 \leq p \leq \infty$, we denote by $L^p = L^p(\mathbb{R})$ the Banach space of Lebesgue measurable functions on \mathbb{R} with the usual norm $\|\cdot\|_{L^p}$. For $s \in \mathbb{R}$, the L^2 -based Sobolev space of order s of complex-valued functions f will be denoted by $H_{\mathbb{C}}^s = H_{\mathbb{C}}^s(\mathbb{R})$ and the usual norm on this space is denoted by $\|\cdot\|_{H^s}$. More generally, if B is any Banach space the norm on B will be denoted by $\|\cdot\|_B$. We denote by $H_{\mathbb{R}}^s = H_{\mathbb{R}}^s(\mathbb{R})$ the space of all real-valued functions f in $H_{\mathbb{C}}^s$ and $H_+^s(\mathbb{R})$ denotes the space of all functions f in $H_{\mathbb{R}}^s$ such that $f(x) > 0$ on \mathbb{R} . If B_1 and B_2 are Banach spaces, then their Cartesian product $B_1 \times B_2$ is a Banach space with a product norm defined by $\|(f, g)\|_{B_1 \times B_2} := \|f\|_{B_1} + \|g\|_{B_2}$. In particular, we define

$$\mathcal{H} = H_{\mathbb{C}}^1 \times H_{\mathbb{C}}^1 \times H_{\mathbb{R}}^1 \quad \text{and} \quad \mathcal{Y} = H_{\mathbb{C}}^1 \times H_{\mathbb{R}}^1 \times H_{\mathbb{R}}^1.$$

If B is a Banach space and G is a subset of B , we say that a sequence $\{x_n\}$ in B converges to G if

$$\lim_{n \rightarrow \infty} \inf_{g \in G} \|x_n - g\|_B = 0.$$

Also, for each $T > 0$, we denote by $\mathcal{C}([0, T]; B)$ the Banach space of continuous maps f from $[0, T]$ to B , with norms given by

$$\|f\|_{\mathcal{C}([0, T]; B)} = \sup_{t \in [0, T]} \|f(t)\|_B.$$

For any $a \in \mathbb{R}$, we denote by T_a the translation operator defined by $(T_a f)(\cdot) = f(\cdot + a)$. Also, we use notation $A_1 \lesssim A_2$ if there exist constants C_1 and C_2 such that $A_1 \leq C_1 A_2$ and $A_2 \leq C_2 A_1$. The symbol C will be used throughout to denote various constants whose exact values are not important and which may differ from one line to the next.

Following standard notations in the literature, D^s and J^s , respectively, denote the multiplication operators (via the Fourier transform) with symbols $|\xi|^s$ and $(1 + \xi^2)^{s/2}$. Thus, J^{-s} is the usual Bessel potential and the classical Sobolev space in the line is defined by $H^s = J^{-s}(L^2)$ with $\|\varphi\|_{H^s} = \|J^s \varphi\|_{L^2}$. Also, throughout the work for any $1 \leq p \leq \infty$ we denote by p' the exponent such that $\frac{1}{p} + \frac{1}{p'} = 1$ and we will use the space-time Lebesgue spaces $L_T^\rho L_x^\nu$ and $L_x^\nu L_T^\rho$ equipped with the norms

$$\begin{aligned} \|f(x, t)\|_{L_T^\rho L_x^\nu} &= \left\| \|f(\cdot, t)\|_{L^\nu(\mathbb{R})} \right\|_{L^\rho([0, T])}, \\ \|f(x, t)\|_{L_x^\nu L_T^\rho} &= \left\| \|f(x, \cdot)\|_{L^\rho([0, T])} \right\|_{L^\nu(\mathbb{R})}. \end{aligned}$$

We use $S(t)$ and $V(t)$ given by

$$S(t) = e^{it\partial_x^2} \quad \text{and} \quad V(t) = e^{-t\partial_x^3}, \quad (2.1)$$

to denote the linear propagators for the Schrödinger and the KdV equations respectively. Given $s, k \in \mathbb{R}$ and $0 < b < 1$, we define two function spaces $X^{s,b}$ and $Y^{k,b}$ as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with respect to the norms

$$\begin{aligned} \|f\|_{X^{s,b}} &:= \iint \langle \xi \rangle^{2s} \langle \tau + \xi^2 \rangle^{2b} |\widehat{f}(\xi, \tau)|^2 d\tau d\xi \\ &= \|S(-t)f\|_{H_t^b(\mathbb{R}; H_x^s)} \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} \|f\|_{Y^{k,b}} &:= \iint \langle \xi \rangle^{2k} \langle \tau - \xi^3 \rangle^{2b} |\widehat{f}(\xi, \tau)|^2 d\tau d\xi \\ &= \|V(-t)f\|_{H_t^b(\mathbb{R}; H_x^k)}, \end{aligned} \quad (2.3)$$

where $\langle \cdot \rangle := 1 + |\cdot|$.

Finally, we introduce the following even smooth cut-off function $\psi \in C_0^\infty(\mathbb{R})$ given by

$$\psi(t) = \begin{cases} 1, & \text{for } |t| \leq 1, \\ 0, & \text{for } |t| \geq 2, \end{cases} \quad (2.4)$$

and define $\psi_T(t) := \psi(\frac{t}{T})$.

We now state our main results.

2.2. Well-posedness results. Here we state the main results about well-posedness theory established in this work for the IVPs associated to the systems (1.1) and (1.2).

Theorem 2.1. *Consider system (1.1) with $1 \leq p = \frac{n_1}{n_2} \in \mathbb{Q}^+$, n_2 odd, and $q_j > 0$ for $j = 1, 2$. For any given data $(u_{10}, u_{20}, v_0) \in H^1 \times H^1 \times H^1$ there is a positive time $T = T(\|u_{10}\|_{H^1}, \|u_{20}\|_{H^1}, \|v_0\|_{H^1})$ and a unique solution (u_1, u_2, v) to the IVP associated to (1.1) such that*

$$(u_1, u_2, v) \in \mathcal{C}([0, T]; H^1 \times H^1 \times H^1) \quad (2.5)$$

$$\|u_1\|_{L_x^2 L_T^\infty} + \|u_2\|_{L_x^2 L_T^\infty} + \|v\|_{L_x^2 L_T^\infty} \lesssim (1 + T)^{\frac{3}{4}+}. \quad (2.6)$$

Moreover, for any $T' < T$ there exists a neighborhood \mathcal{V} of (u_{10}, u_{20}, v_0) in $H^1 \times H^1 \times H^1$ such that the map

$$(\tilde{u}_{10}, \tilde{u}_{20}, \tilde{v}_0) \longmapsto (\tilde{u}_1, \tilde{u}_2, \tilde{v})$$

from \mathcal{V} into the class defined by (2.5)-(2.6) with T' instead of T is continuous. Also, for $1 \leq p < \frac{4}{3}$ and q_j for $j = 1, 2$, verifying

$$0 < q_j < \begin{cases} 4 & \text{if } \tau_j > 0, \\ \infty & \text{if } \tau_j \leq 0, \end{cases} \quad (2.7)$$

the local solution can be extended to any time interval $[0, T]$ with T arbitrary large.

Theorem 2.2. Consider system (1.2) with $q > 0$ and $1 \leq p_j = \frac{n_{j1}}{n_{j2}} \in \mathbb{Q}^+$, n_{j2} odd, for $j = 1, 2$. For any given data $(u, v_{10}, v_{20}) \in H^1 \times H^1 \times H^1$ there is a positive time $T = T(\|u_0\|_{H^1}, \|v_{10}\|_{H^1}, \|v_{20}\|_{H^1})$ and a unique solution (u, v_1, v_2) to the IVP associated to (1.2) such that

$$(u, v_1, v_2) \in \mathcal{C}([0, T]; H^1 \times H^1 \times H^1) \quad (2.8)$$

$$\|u\|_{L_x^2 L_T^\infty} + \|v_1\|_{L_x^2 L_T^\infty} + \|v_2\|_{L_x^2 L_T^\infty} \lesssim (1 + T)^{\frac{3}{4}+}. \quad (2.9)$$

Moreover, for any $T' < T$ there exists a neighborhood \mathcal{V} of (u_0, v_{10}, v_{20}) in $H^1 \times H^1 \times H^1$ such that the map

$$(\tilde{u}_0, \tilde{v}_{10}, \tilde{v}_{20}) \longmapsto (\tilde{u}, \tilde{v}_1, \tilde{v}_2)$$

from \mathcal{V} into the class defined by (2.8)-(2.9) with T' instead of T is continuous. Also, for $1 \leq p_j < \frac{4}{3}$, $j = 1, 2$, and q verifying

$$0 < q < \begin{cases} 4 & \text{if } a > 0, \\ \infty & \text{if } a \leq 0, \end{cases} \quad (2.10)$$

the local solutions can be extended to any time interval $[0, T]$ with T arbitrary large.

As discussed in the introduction, for the special cases $q_1 = q_2 = 2p = 2$ for system (1.1) and $2p_1 = 2p_2 = q = 2$ for system (1.2) we prove the following more general local well-posedness results using contraction mapping principle in the framework of Bourgain's spaces. More precisely, in these cases we have the following local well-posedness theorems.

Theorem 2.3. Consider system (1.1) with $q_1 = q_2 = 2p = 2$. Let (u_{10}, u_{20}, v_0) belonging to the space $H^{s_1} \times H^{s_2} \times H^k$ with $k > -3/4$ provided:

$$(a) \max\{k - 1, \kappa/4\} < s_j < \kappa + 2 \text{ if } \gamma_j = 0, \text{ for } j = 1, 2,$$

$$(b) \max\{k - 1, k/4\} < s_j < \kappa + 2 \text{ and } s_j \geq 0 \text{ if } \gamma_j \neq 0, \text{ for } j = 1, 2.$$

Then, there exist a time $T(\|u_{10}\|_{H^{s_1}}, \|u_{20}\|_{H^{s_2}}, \|v_0\|_{H^k}) > 0$ and a unique solution for the integral equations associated to the IVP for (1.1) in an appropriate Bourgain's space contained in $\mathcal{C}([0, T]; H^{s_1} \times H^{s_2} \times H^k)$.

Moreover, the mapping $(u_{10}, u_{20}, v_0) \longmapsto (u_1(\cdot, t), u_2(\cdot, t), v(\cdot, t))$ is locally Lipschitz.

Theorem 2.4. Consider system (1.2) with $2p_1 = 2p_2 = q = 2$. Let (u_0, v_{10}, v_{20}) belonging to the space $H^s \times H^{k_1} \times H^{k_2}$ with $k_j > -3/4$, $j = 1, 2$, provided:

$$(a) s - 2 \leq k_j < \min\{4s, s + 1\} \text{ if } \gamma = 0, \text{ for } j = 1, 2,$$

$$(b) s - 2 \leq k_j < \min\{4s, s + 1\} \text{ and } s \geq 0 \text{ if } \gamma \neq 0, \text{ for } j = 1, 2.$$

Then, there exist a positive time $T(\|u_0\|_{H^s}, \|v_{10}\|_{H^{k_1}}, \|v_{20}\|_{H^{k_2}})$ and a unique solution for the integral equations associated to the IVP for (1.2) in an appropriate Bourgain's space contained in $\mathcal{C}([0, T]; H^s \times H^{k_1} \times H^{k_2})$.

Moreover, the mapping $(u_0, v_{10}, v_{20}) \longmapsto (u(\cdot, t), v_1(\cdot, t), v_2(\cdot, t))$ is locally Lipschitz.

2.3. Existence and stability results. To state our existence theorem, let us denote by $\mathcal{O}_{r,l,m}$ the set of all normalized solutions (ϕ_1, ϕ_2, w) of (1.9) satisfying the condition (1.11). We assume that the following conditions hold:

$$\begin{cases} \gamma_1, \gamma_2 > 0, \beta > 0, \alpha_1, \alpha_2 > 0, \\ 0 < q_1, q_2 < 4, \text{ and } p = \frac{n_1}{n_2} \in \mathbb{Q}^+, n_2 \text{ odd.} \end{cases} \quad (2.11)$$

The following theorem guarantees that the set $\mathcal{O}_{r,l,m}$ is non-empty.

Theorem 2.5. *Suppose the assumptions (2.11) hold and that $0 < p < 4$. Then for every $(r, l, m) \in \mathbb{R}_+^3$, there exists a solution*

$$(\sigma_{1r}, \sigma_{2l}, c_m, \phi_r, \phi_l, w_m) \in \mathbb{R}_+^2 \times \mathbb{R} \times \mathcal{H}$$

to the system (1.9) satisfying the condition

$$\|\phi_r\|_{L^2}^2 = r, \|\phi_l\|_{L^2}^2 = l, \text{ and } \|w_m\|_{L^2}^2 = m \quad (2.12)$$

with σ_{1r}, σ_{2l} , and c_m being the Lagrange multipliers. Moreover, $w_m(x) > 0$ for all $x \in \mathbb{R}$ and there exists $(\zeta_j, R_j) \in S^1 \times H_+^1(\mathbb{R})$ such that

$$\phi_r(x) = \zeta_1 R_1(x) \text{ and } \phi_l(x) = \zeta_2 R_2(x), \text{ for all } x \in \mathbb{R}.$$

In particular, $(\sigma_{1r}, \sigma_{2l}, c_m, R_1, R_2, w_m)$ is a real-valued positive solution of (1.9).

Our approach to study the stability of solitary waves is purely variational. For any $(r, l, m) \in \mathbb{R}_+^2 \times \mathbb{R}$, we consider a new variational formulation of solitary waves, namely the problem of minimizing the functional $E(h_1, h_2, g)$ over the set

$$\Pi_{r,l,m} = \{\Delta \in \mathcal{H} : \Delta = (h_1, h_2, g), Q(h_1) = r, Q(h_2) = l, \text{ and } H(\Delta) = m\}. \quad (2.13)$$

The family of minimization problems

$$\Lambda(r, l, m) = \inf\{E(h_1, h_2, g) : (h_1, h_2, g) \in \Pi_{r,l,m}\} \quad (2.14)$$

is suitable for studying the stability properties of travelling solitary waves because both E and the constraint functionals $Q(h_j)$ and $H(h_1, h_2, g)$ are invariants of motion of (1.1). For such a variational problem, an easy consequence of application of the concentration compactness argument [23, 10] is that the set of global minimizers forms a stable set for the associated initial-value problem, in that a solution which is initially close to this set will remain close to it for later times.

The next result concerns the existence of solutions to the variational problem (2.14) and their relation with those in $\mathcal{O}_{r,l,m}$.

Theorem 2.6. *Suppose the assumptions (2.11) hold and that $1 \leq p < 4/3$. Then*

(i) *every minimizing sequence $\{(h_{1n}, h_{2n}, g_n)\}_{n \geq 1}$ for $\Lambda(r, l, m)$ enjoys the following compactness property: there exists a subsequence $\{(h_{1n_k}, h_{2n_k}, g_{n_k})\}_{k \geq 1}$, a family $(y_k) \subset \mathbb{R}$, and a function $(\Phi_1, \Phi_2, w) \in \mathcal{H}$ such that the translated subsequence*

$$\{(T_{y_k} h_{1n_k}, T_{y_k} h_{2n_k}, T_{y_k} g_{n_k})\}_{k \geq 1}$$

converges strongly to (Φ_1, Φ_2, w) in \mathcal{H} . The function (Φ_1, Φ_2, w) achieves the minimum,

$$(\Phi_1, \Phi_2, w) \in \Pi_{r,l,m} \text{ and } E(\Phi_1, \Phi_2, w) = \Lambda(r, l, m).$$

(ii) if (Φ_1, Φ_2, w) is a solution of (2.14) and $\|w\|_{L^2}^2 = N$, then there exists $(\zeta_j, \phi_j) \in S^1 \times H_+^1(\mathbb{R})$ such that $(\phi_1, \phi_2, w) \in \mathcal{O}_{r,l,N}$ and

$$\Phi_1(x) = \zeta_1 e^{-ib_{r,l,m}(N)x} \phi_1(x) \quad \text{and} \quad \Phi_2(x) = \zeta_2 e^{-ib_{r,l,m}(N)x} \phi_2(x), \quad x \in \mathbb{R},$$

where $b_{r,l,m}(A)$ is defined by

$$b_{r,l,m}(A) = \frac{m-A}{2r+2l} \text{ for any } A \geq 0. \quad (2.15)$$

Furthermore, if $\gamma_1 = \gamma_2 = 0$, then the function w can be chosen to be strictly positive on \mathbb{R} .

We use the following notion of stability.

Definition 2.7. For $(r, l, m) \in \mathbb{R}_+^2 \times \mathbb{R}$, let $\mathcal{P}_{r,l,m}$ be the set of solutions of (2.14). We say that the set $\mathcal{P}_{r,l,m}$ of solitary-wave profiles is stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that for any initial datum $\Delta_0 = (h_{01}, h_{02}, g_0)$ satisfying

$$\Delta_0 \in \mathcal{H}, \quad \inf \{ \|(\Phi_1, \Phi_2, w) - \Delta_0\|_{\mathcal{H}} : (\Phi_1, \Phi_2, w) \in \mathcal{P}_{r,l,m} \} < \delta,$$

then the solution $\Delta(t, x) = (u_1(t, x), u_2(t, x), v(t, x))$ of (1.1) satisfies for all times $t \geq 0$,

$$\inf \{ \|(\Phi_1, \Phi_2, w) - \Delta(t, \cdot)\|_{\mathcal{H}} : (\Phi_1, \Phi_2, w) \in \mathcal{P}_{r,l,m} \} < \epsilon.$$

Our stability result reads as follows.

Theorem 2.8. Suppose the assumptions (2.11) hold and that $1 \leq p < 4/3$. Then the following statements hold:

- (i) every minimizing sequence $\{(f_{1n}, f_{2n}, g_n)\}$ for (2.14) converges to $\mathcal{P}_{r,l,m}$ in \mathcal{H} .
- (ii) the set $\mathcal{P}_{r,l,m}$ of minimizers is stable in the sense as in Definition 2.7.
- (iii) the set $\mathcal{P}_{r,l,m}$ of minimizers forms a true-three parameter family, that is, if the sets $\mathcal{P}_{r_i, l_i, m_i}$ contain $(\Phi_1^{(i)}, \Phi_2^{(i)}, w^{(i)})$ for $(r_1, l_1, m_1) \neq (r_2, l_2, m_2)$, then

$$(\Phi_1^{(1)}, \Phi_2^{(1)}, w^{(1)}) \neq (\Phi_1^{(2)}, \Phi_2^{(2)}, w^{(2)}).$$

Remark 2.9. Our stability result generalizes and extends analogous results for (1+1)-component NLS-KdV solitary waves previously obtained in [11], which considered the particular case $u_2 \equiv 0, \gamma_1 = 0, p = 1$, and $\alpha_1 = 1/6$; [1], which studied the case when $u_2 \equiv 0, \gamma_1 = 0, p = 1$, and α_1 in some neighborhood of $1/6$; [2], which considered the case when $u_2 \equiv 0, \gamma_1 > 0, 1 \leq q_1 < 4, p = 1$, and $\alpha_1 > 0$; and of [3], which proved a stability result for certain sets of solitary waves in the special case when $u_2 \equiv 0, \gamma_1 = 0, p = 1, \alpha_1 > 0$, and the wavespeed σ_1 is near $c^2/4$.

Remark 2.10. Implicit in the notion of stability above is the assumption that (2+1)-component NLS-gKdV is globally well-posed in the energy space \mathcal{H} . Our global well-posedness theory requires the restriction $1 \leq p < 4/3$. As far as we know, it remains an open question whether (1.1) is well-posed in \mathcal{H} for $0 < p < 1$. If one assumes that the (2+1)-component NLS-gKdV is globally well-posed in \mathcal{H} for the range $0 < p < 4/3$, then the conclusions of Theorems 2.6 and 2.8 continue to hold for these values of p .

Remark 2.11. Although we do not pursue these topics here, similar problems related to (1+2)-component NLS-gKdV system are that of the existence and stability results concerning nontrivial solitary wave solutions of the form

$$(e^{i\lambda t} e^{i\sigma(x-\sigma t)/2} \phi(x-\sigma t), w_1(x-\sigma t), w_2(x-\sigma t)),$$

where the functions $\phi : \mathbb{R} \rightarrow \mathbb{C}$, $w_1, w_2 : \mathbb{R} \rightarrow \mathbb{R}$ vanish at $\pm\infty$, and the parameters λ and σ are real. As in the (2+1)-component case, one can study the existence and stability questions of (1+2)-component NLS-gKdV solitary waves via their variational characterizations. To find a true two-parameter family of travelling solitary wave solutions (parameterized by σ and λ), one considers the two-parameter variational problem

$$\inf \left\{ K(U) : U = (f, g_1, g_2) \in \mathcal{Y} : \|f\|_{L^2}^2 = l > 0 \text{ and } \sum_{j=1}^2 \|g_j\|_{L^2}^2 = r > 0 \right\}. \quad (\text{P2})$$

As usual in the method of concentration compactness, putting the method into practice requires verifying the strict subadditivity condition for the function defined by (P2) with respect to the constraint variables (see Lemma 4.7 below). If one can prove the strict subadditivity inequality, all of what is proved in Section 4 below should be readily extendable to study the problem (P2), in which case an analogous result of Theorem 2.5 concerning the existence of minimizers will hold for the problem (P2) as well. To obtain stability properties of solitary waves, one then considers the problem of finding for any $(l, r) \in (0, \infty) \times \mathbb{R}$,

$$\Lambda_{\text{NKK}}(l, r) = \inf \{ K(U) : U = (h, g_1, g_2) \in \mathcal{Y}, Q(h) = l, \text{ and } G(U) = r \}. \quad (2.16)$$

The family of problems (2.16) is suitable for studying the stability properties of solitary waves for (1.2) because both $K(U)$ and the constraint functionals $Q(h)$ and $G(U)$ are invariants of motion of (1.2). Once the existence of minimizers for the problem (P2) is guaranteed, one can follow the same arguments as in Section 5 below to obtain the stability result for the (1+2)-component NLS-gKdV solitary waves.

3. Local Well-Posedness in the Energy Regularity and Below

In this section we supply proofs of the local-well-posedness results to the IVPs associated to the (2+1)-component and (1+2)-component NLS-KdV systems. We provide details of the proof of the Theorem 2.1 only, because the proof of the Theorem 2.2 follows similarly.

3.1. Preliminary estimates. Here we recall some important smoothing properties related to the free propagators $S(t) = e^{it\partial_x^2}$ and $V(t) = e^{-t\partial_x^3}$, which will be useful to construct the local solutions in the energy space.

We begin by recalling the low-high projections operators via dyadic decomposition in the line. Let η be a smooth nonnegative function such that

$$\eta(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1, \\ 0 & \text{if } |\xi| \geq 2 \end{cases}$$

with $\widehat{\phi}(\xi) = \eta(\xi) - \eta(2\xi) \in \mathcal{S}(\mathbb{R})$, supported in the set $\{\xi; 1/2 \leq |\xi| \leq 2\}$, and satisfying $\sum_{j \in \mathbb{Z}} \widehat{\phi}(\xi/2^j) = 1$, $\xi \neq 0$. Define ϕ_l by

$$\widehat{\phi}_l(\xi) = 1 - \sum_{j=1}^{\infty} \widehat{\phi}\left(\frac{\xi}{2^j}\right) = \sum_{j=0}^{\infty} \widehat{\phi}(2^j \xi)$$

and denote by P_l and P_h the operators

$$P_l f = \phi_l * f \quad \text{and} \quad P_h f = (I - P_l)f. \quad (3.1)$$

Proposition 3.1. *The operators $P_l f$ and P_h satisfy the following properties:*

- (a) $\|P_l D^\alpha f\|_{L^\nu} \lesssim \|f\|_{L^2}$ for all $f \in \mathcal{S}(\mathbb{R})$, $\alpha \geq 0$ and $2 \leq \nu < \infty$.
- (b) $\|P_h g\|_{L_x^\nu L_T^\rho} \lesssim \|g\|_{L_x^\nu L_T^\rho}$ for all $g \in \mathcal{S}(\mathbb{R}^2)$ and $1 \leq \nu, \rho \leq \infty$.

Proof. Since $\widehat{\phi}_l(\xi) = 0$ for $\xi \geq 2$, the first estimate follows by using Sobolev's embedding and Plancherel's theorems. Indeed,

$$\|P_l D^\alpha f\|_{L^\nu} \lesssim \|D^{\frac{1}{2}-\frac{1}{\nu}} P_l D^\alpha f\|_{L^2} = \|\widehat{\phi}_l(\xi) |\xi|^{\alpha+\frac{1}{2}-\frac{1}{\nu}} \widehat{f}(\xi)\|_{L^2} \lesssim \|f\|_{L^2}.$$

The estimate in (b) can be found in [26]. \square

For any unitary group $\{W(t)\}_{t \in \mathbb{R}}$ in $L^2(\mathbb{R})$ we have the following retarded convolution estimate

$$\left\| \int_0^t W(t-t') f(\cdot, t') dt' \right\|_{L_T^\infty L_x^2} \leq \|f\|_{L_T^1 L_x^2}. \quad (3.2)$$

Indeed, by Minkowski inequality and the properties of W we get

$$\left\| \int_0^t W(t-t') f(\cdot, t') dt' \right\|_{L_x^2} \leq \int_0^t \|f(\cdot, t')\|_{L_x^2} dt'.$$

Therefore,

$$\left\| \int_0^t W(t-t') f(\cdot, t') dt' \right\|_{L_T^\infty L_x^2} \leq \sup_{0 \leq t \leq T} \int_0^t \|f(\cdot, t')\|_{L_x^2} dt' \leq \|f\|_{L_T^1 L_x^2}.$$

Obviously, $S(t)$ and $V(t)$ satisfy (3.2).

Lemma 3.2 (Smoothing effects for $e^{it\partial_x^2}$). *Let $\mu > 3/4$ and $s > 1/2$. Then, for all positive T we have the following maximal function type estimates*

- (a) $\|S(t)\varphi\|_{L_x^2 L_T^\infty} \leq c(1+T)^\mu \|\varphi\|_{H^s}$.
- (b) $\left\| \int_0^t S(t-t') f(\cdot, t') dt' \right\|_{L_x^2 L_T^\infty} \leq c(1+T)^\mu \|J_x^s f\|_{L_T^1 L_x^2}.$

All constants c are independent of the time T .

Proof. For the proof of estimate in (a) see Corollary 2.9 in [21]. The estimate in (b) is obtained as follows:

$$\begin{aligned}
\left\| \int_0^t S(t-t')f(\cdot, t') dt' \right\|_{L_x^2 L_T^\infty} &\leq \int_0^T \|S(t)S(-t')f(\cdot, t')\|_{L_x^2 L_T^\infty} dt' \\
&\leq c \int_0^T (1+T)^\mu \|J_x^s S(-t')f(\cdot, t')\|_{L_x^2} dt' \\
&\leq c \int_0^T (1+T)^\mu \|J_x^s f(\cdot, t')\|_{L_x^2} dt' \\
&= c(1+T)^\mu \|J_x^s f\|_{L_T^1 L_x^2},
\end{aligned} \tag{3.3}$$

and we finished the proof. \square

The following result tells about similar results for the unitary group $V(t)$.

Lemma 3.3 (Smoothing effects for $e^{-t\partial_x^3}$). *Let μ and $s > 3/4$. Then, for all positive T the following maximal function type estimates hold true.*

$$(a) \quad \|V(t)\varphi\|_{L_x^2 L_T^\infty} \leq c(1+T)^\mu \|\varphi\|_{H^s}.$$

$$(b) \quad \left\| \int_0^t V(t-t')g(\cdot, t') dt' \right\|_{L_x^2 L_T^\infty} \leq c(1+T)^\mu \|J_x^s g\|_{L_T^1 L_x^2}.$$

Also, we have

$$(c) \quad \left\| \partial_x \int_0^t V(t-t')g(\cdot, t') dt' \right\|_{L_T^\infty L_x^2} \leq c\|g\|_{L_x^1 L_T^2}.$$

All constants c are independent of the time T .

Proof. For the proof of estimate (a) see Corollary 2.9 in [21] and (b) is obtained similarly to (3.3). The estimate in (c) is the classical dual version of the Kato type smoothing effect

$$\|\partial_x V(t)\varphi\|_{L_x^\infty L_t^2} \leq c\|\varphi\|_{L^2}, \tag{3.4}$$

proved in Lemma 2.1 of [21]. \square

Another important ingredient is the Proposition 2.7 of [25], which establishes a version of the so-called Christ-Kiselev lemma. The result reads as follows:

Proposition 3.4. *Let $s_1, s_2 \in \mathbb{R}$ and $1 \leq \rho_1, \nu_1, \rho_2, \nu_2 \leq \infty$ such that for all $\varphi \in \mathcal{S}(\mathbb{R})$,*

$$\|D_x^{s_1} V(t)\varphi\|_{L_x^{\nu_1} L_T^{\rho_1}} \lesssim \alpha_T \|\varphi\|_{L^2}, \tag{3.5}$$

$$\|D_x^{s_2} V(t)\varphi\|_{L_x^{\nu_2} L_T^{\rho_2}} \lesssim \beta_T \|\varphi\|_{L^2}, \tag{3.6}$$

where α_T, β_T are positive constants depending on T . Then for all $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\left\| D_x^{s_2} \int_0^t V(t-t') f(x, t') dt' \right\|_{L_T^\infty L_x^2} \lesssim \beta_T \|f\|_{L_x^{\nu'_2} L_T^{\rho'_2}}, \quad (3.7)$$

$$\left\| D_x^{s_1+s_2} \int_0^t V(t-t') f(x, t') dt' \right\|_{L_x^{\nu_1} L_T^{\rho_1}} \lesssim \alpha_T \beta_T \|f\|_{L_x^{\nu'_2} L_T^{\rho'_2}}, \quad (3.8)$$

provided the conditions

$$\min(\nu_1, \rho_1) > \max(\nu'_2, \rho'_2) \quad \text{or} \quad \rho_1 = \infty \quad \text{and} \quad \nu'_2, \rho'_2 < \infty. \quad (3.9)$$

The following result is concerned to the localized maximal function type estimate for $V(t)$ in high frequencies and its proof can be found in [16]. Here we sketch the proof with only one small difference.

Lemma 3.5. *Let $\mu > 3/4$. Then for all $g \in \mathcal{S}(\mathbb{R}^2)$, we have*

$$\left\| \int_0^t V(t-t') P_{\mathbf{h}} g(x, t') dt' \right\|_{L_x^2 L_T^\infty} \lesssim (1+T)^\mu \|P_{\mathbf{h}} g\|_{L_x^1 L_T^2}.$$

Proof. From the definition of $P_{\mathbf{h}}$ it follows that $\|J_x^s P_{\mathbf{h}} \varphi\|_{L_x^2} \lesssim \|D_x^s P_{\mathbf{h}} \varphi\|_{L_x^2}$ for all $s \in \mathbb{R}$ and $\varphi \in \mathcal{S}(\mathbb{R})$. So, taking $s > 3/4$, by Lemma 3.3-(a) and previous inequality we have

$$\|V(t) P_{\mathbf{h}} \varphi\|_{L_x^2 L_T^\infty} \lesssim (1+T)^\mu \|J_x^s P_{\mathbf{h}} \varphi\|_{L_x^2} \lesssim (1+T)^\mu \|D_x^s P_{\mathbf{h}} \varphi\|_{L_x^2},$$

which implies

$$\|D^{-1} V(t) P_{\mathbf{h}} \varphi\|_{L_x^2 L_T^\infty} \lesssim (1+T)^\mu \|P_{\mathbf{h}} \varphi\|_{L_x^2}, \quad (3.10)$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$.

On the other hand, by (3.4)

$$\|D_x^1 V(t) P_{\mathbf{h}} \varphi\|_{L_x^\infty L_T^2} \lesssim \|P_{\mathbf{h}} \varphi\|_{L_x^2}. \quad (3.11)$$

Finally, applying Proposition 3.4 with $s_1 = -1$, $s_2 = 1$, $(\nu_1, \rho_1) = (2, \infty)$ and $(\nu_2, \rho_2) = (\infty, 2)$ we obtain the desired result. \square

Finally, we recall the following particular version of the Gagliardo-Nirenberg inequality.

Proposition 3.6. *Let $p \geq 2$. Then, for any $f \in H^1(\mathbb{R})$ it holds that*

$$\|f\|_{L^p} \leq c_p \|f\|_{L^2}^{\frac{1}{2} + \frac{1}{p}} \|\partial_x f\|_{L^2}^{\frac{1}{2} - \frac{1}{p}} \leq c_p \|f\|_{H^1}.$$

We finish this subsection by recording some more linear and nonlinear estimates in the framework of the Bourgain space. These estimates are used to get well-posedness theory below energy space for some particular nonlinearities involved in the systems (1.1) and (1.2). To simplify the exposition, we borrow notations from [15].

Let $W_\phi(t) := e^{-it(-i\partial_x)}$ represents the unitary group that describes the solution of the linear problem

$$i\partial_t w - \phi(-i\partial_x)w = 0. \quad (3.12)$$

With this notation, let us define the associated function space $X^{s,b}(\phi)$, $s, b \in \mathbb{R}$ as the completion of the Schwartz space with respect to the norm

$$\|f\|_{X^{s,b}(\phi)} := \|W_\phi(-t)f\|_{H_t^b(\mathbb{R}; H_x^s(\mathbb{R}))} = \|\langle \xi \rangle^s \langle \tau + \phi(\xi) \rangle^b \widehat{f}(\tau, \xi)\|_{L_\tau^2 L_\xi^2}. \quad (3.13)$$

Considering respectively $\phi(\xi) = \phi_1(\xi) = \xi^2$ and $\phi(\xi) = \phi_2(\xi) = -\xi^3$ we have $X^{s,b}(\phi_1) = X^{s,b}$ and $X^{k,b}(\phi_2) = Y^{k,b}$.

Lemma 3.7. *Let $0 \leq T \leq 1$ and $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$. Then for any $f \in H^s$ and $g \in X^{s,b'}(\phi)$, the following estimates hold true.*

$$\|\psi(t)W_\phi(t)f\|_{X^{s,b}(\phi)} \lesssim \|f\|_{H^s}, \quad (3.14)$$

$$\|\psi_T(t) \int_0^t W_\phi(t-t')g(t', \cdot)dt'\|_{X^{s,b}(\phi)} \lesssim T^{1-b+b'} \|g\|_{X^{s,b'}(\phi)}. \quad (3.15)$$

Proof. Proof of this lemma can be found in [18], so we omit the details. \square

Now we state the following bilinear estimate whose proof can be found in [22].

Lemma 3.8. *Let $\kappa > -\frac{3}{4}$, $b, b' = \frac{1}{2}+$, then for any $v, w \in Y^{\kappa,b}$*

$$\|\partial_x(vw)\|_{Y^{\kappa,b'-1}} \lesssim \|v\|_{Y^{\kappa,b}} \|w\|_{Y^{\kappa,b}}. \quad (3.16)$$

Also, we record the following trilinear estimate from [5].

Lemma 3.9. *Let $s \geq 0$, $\frac{1}{2} < b < 1$ and $u \in H^{s,b}$. Then for any $a \geq 0$, one has*

$$\| \|u\|^2 u \|_{H^{s,-a}} \leq C \|u\|_{X^{s,b}}^3. \quad (3.17)$$

The following results are obtained in [30].

Lemma 3.10. *Let $\kappa \geq -1$ and suppose $s - \kappa \leq 2$ when $s \geq 0$, $s + \kappa \geq -2$ when $s < 0$. Also consider $c, c' = \frac{1}{2}+$. Then for any $u \in X^{s,c}$ and $v \in Y^{\kappa,b}$, one has*

$$\|uv\|_{X^{s,c'-1}} \lesssim \|u\|_{X^{s,c}} \|v\|_{Y^{\kappa,b}}. \quad (3.18)$$

Lemma 3.11. *Let $s > -\frac{1}{4}$, $\kappa < 4s$ and consider $\kappa - s < 1$ when $s \geq 0$, $\kappa - 2s < 1$ when $s < 0$. Also, take $b', c = \frac{1}{2}+$. Then for any $u_1, u_2 \in X^{s,c}$*

$$\|\partial_x(u_1 \bar{u}_2)\|_{Y^{\kappa,b'-1}} \lesssim \|u_1\|_{X^{s,c}} \|u_2\|_{X^{s,c}}. \quad (3.19)$$

3.2. Local and global theory for (2+1)-component NLS-gKdV. Throughout this section we assume the following conditions on the indices of the nonlinearities

$$q_j > 0 \ (j = 1, 2) \quad \text{and} \quad p = \frac{n_1}{n_2} \geq 1 \quad \text{with} \quad n_1, n_2 \in \mathbb{N} \quad \text{and} \quad n_2 \text{ odd}. \quad (3.20)$$

Using Duhamel's formula, we consider the IVP associated to the system (1.1) in the equivalent system of integral equations

$$\begin{cases} u_j(t) = S(t)u_{j0} + i \int_0^t S(t-t') [\alpha_j u_j v + \gamma_j |u_j|^{q_j} u] (t') dt' \quad (j = 1, 2), \\ v(t) = V(t)v_0 - \int_0^t V(t-t') [\beta v^p \partial_x v + \partial_x (\alpha_1 |u_1|^2 + \alpha_2 |u_2|^2)] (t') dt'. \end{cases} \quad (3.21)$$

Our goal is to solve (3.21) by applying the contraction mapping principle in a suitable subspace of the continuous functions $\mathcal{C}([0, T]; H^1 \times H^1 \times H^1)$. The main tool in the proof is the use of a localized maximal function, described in Lemma 3.5, in order to estimate the non-homogeneous term

$$\int_0^t V(t-t')g(x, t')dt',$$

with a good control for the terms $\partial_x(|u_j|^2)$.

Remark 3.12. We note that our proof include and extend the results given by Guo and Miao in [19], for energy regularity, in the context of the classical (1+1)-component NLS-KdV system ($p = 1$ and $\gamma_j = 0$). In fact, here we contemplate fractional powers for KdV component and also we emphasize that our fixed-point procedure is developed in a different environment than the one used in [19].

Proof of Theorem 2.1 (local theory). Let $\mu > 3/4$. For positive numbers T , M_1, M_2 and M , to be chosen later, we define a function space

$$\mathcal{Z}_T = \left\{ (u_1, u_2, v) \in \mathcal{C}([0, T]; H^1 \times H^1 \times H^1); \|u_j\|_{\mathcal{S}_T} \leq M_j \text{ and } \|v\|_{\mathcal{K}_T} \leq M \right\},$$

where

$$\|u_j\|_{\mathcal{S}_T} := \|J_x^1 u_j\|_{L_T^\infty L_x^2} + (1+T)^{-\mu} \|u_j\|_{L_x^2 L_T^\infty}, \quad j = 1, 2, \quad (3.22)$$

and

$$\|v\|_{\mathcal{K}_T} := \|J_x^1 v\|_{L_T^\infty L_x^2} + (1+T)^{-\mu} \|v\|_{L_x^2 L_T^\infty}. \quad (3.23)$$

We equip the space \mathcal{Z}_T with the norm defined by

$$\|(u_1, u_2, v)\|_{\mathcal{Z}_T} := \|u_1\|_{\mathcal{S}_T} + \|u_2\|_{\mathcal{S}_T} + \|v\|_{\mathcal{K}_T}. \quad (3.24)$$

It is easy to check that \mathcal{Z}_T is a complete metric space with respect to the norm defined in (3.24).

For $\Delta := (u_1, u_2, v) \in \mathcal{Z}_T$ and with $j = 1, 2$, we define the operators

$$\begin{cases} \Phi_j(\Delta) = S(t)u_{j0} + i \int_0^t S(t-t') [\alpha_j(u_j v) + \gamma_j |u_j|^{q_j} u_j] dt', \\ \Psi(\Delta) = V(t)v_0 - \int_0^t V(t-t') [\beta v^p \partial_x v + \partial_x (\alpha_1 |u_1|^2 + \alpha_2 |u_2|^2)] dt'. \end{cases} \quad (3.25)$$

Claim: $(\Phi_1, \Phi_2, \Psi)(\mathcal{Z}_T) \subset \mathcal{Z}_T$ for a suitable positive time T .

In what follows we consider $\Delta \in \mathcal{Z}_T$.

Estimates for Φ_j . Using (3.2), Lemma 3.2 with $s = 1$, the algebra structure of $H^1(\mathbb{R})$ and Hölder's inequality it follows that

$$\begin{aligned} \|\Phi_j(\Delta)\|_{\mathcal{S}_T} &= \|J_x^1 \Phi_j(\Delta)\|_{L_T^\infty L_x^2} + (1+T)^{-\mu} \|\Phi_j(\Delta)\|_{L_x^2 L_T^\infty} \\ &\lesssim \|u_{j0}\|_{H^1} + \|J_x^1(u_j v)\|_{L_T^1 L_x^2} + \|J_x^1(|u_j|^{q_j} u_j)\|_{L_T^1 L_x^2} \\ &\lesssim \|u_{j0}\|_{H^1} + T \|u_j\|_{L_T^\infty H_x^1} \|v\|_{L_T^\infty H_x^1} + \|J_x^1(|u_j|^{q_j} u_j)\|_{L_T^1 L_x^2}. \end{aligned} \quad (3.26)$$

On the other hand, using the Gagliardo-Nirenberg inequality from Lemma 3.6, one can obtain

$$\begin{aligned}
\|J_x^1(|u_j|^{q_j} u_j)\|_{L_T^1 L_x^2} &= \int_0^T \| |u_j|^{q_j} u_j \|_{L_x^2} dt + \int_0^T \| \partial_x (|u_j|^{q_j} u_j) \|_{L_x^2} dt \\
&\leq \int_0^T \| u_j \|_{L_x^{2(q_j+1)}}^{q_j+1} dt + (q_j + 1) \int_0^T \| |u_j|^{q_j} \partial_x u_j \|_{L_x^2} dt \\
&\lesssim T \| u_j \|_{L_T^\infty H_x^1}^{q_j+1} + (q_j + 1) \int_0^T \| u_j \|_{L_x^\infty}^{q_j} \| \partial_x u_j \|_{L_x^2} dt \\
&\lesssim T \| u_j \|_{L_T^\infty H_x^1}^{q_j+1}.
\end{aligned} \tag{3.27}$$

Thus, combining (3.26) and (3.27) there are positive constants C_0 and C_j such that

$$\begin{aligned}
\| \Phi_j(\Delta) \|_{\mathcal{S}_T} &\leq C_0 \| u_{j0} \|_{H^1} + C_j T \left(\| u_j \|_{L_T^\infty H_x^1} \| v \|_{L_T^\infty H_x^1} + \| u_j \|_{L_T^\infty H_x^1}^{q_j+1} \right) \\
&\leq C_0 \| u_{j0} \|_{H^1} + C_j T \left(\| u_j \|_{\mathcal{S}_T} \| v \|_{\mathcal{K}_T} + \| u_j \|_{\mathcal{S}_T}^{q_j+1} \right) \\
&\leq C_0 \| u_{j0} \|_{H^1} + C_j T (M_j M + M_j^{q_j+1}),
\end{aligned} \tag{3.28}$$

with C_j depending on the parameters α_j, γ_j and q_j .

Estimates for Ψ . The estimate (3.2) combined with the Hölder's inequality and the Sobolev embedding yields

$$\begin{aligned}
\| \Psi(\Delta) \|_{L_T^\infty L_x^2} &\lesssim \| v_0 \|_{L^2} + \| \partial_x (|u_1|^2) + \partial_x (|u_2|^2) + v^p \partial_x v \|_{L_T^1 L_x^2} \\
&\lesssim \| v_0 \|_{L^2} + T \left(\sum_{j=1}^2 \| u_j \|_{L_T^\infty L_x^\infty} \| \partial_x u_j \|_{L_T^\infty L_x^2} + \| v^p \|_{L_T^\infty L_x^\infty} \| \partial_x v \|_{L_T^\infty L_x^2} \right) \\
&\lesssim \| v_0 \|_{L^2} + T \left(\| u_1 \|_{L_T^\infty H_x^1}^2 + \| u_2 \|_{L_T^\infty H_x^1}^2 + \| v \|_{L_T^\infty H_x^1}^{p+1} \right) \\
&\lesssim \| v_0 \|_{L^2} + T \left(\| u_1 \|_{\mathcal{S}_T}^2 + \| u_2 \|_{\mathcal{S}_T}^2 + \| v \|_{\mathcal{K}_T}^{p+1} \right)
\end{aligned} \tag{3.29}$$

In order to estimate the derivative of $\Psi(\vec{w})$ we use Lemma 3.3-(c), to obtain

$$\| \partial_x \Psi(\Delta) \|_{L_T^\infty L_x^2} \lesssim \| v_0 \|_{H^1} + \| v^p \partial_x v + \partial_x (|u_1|^2) + \partial_x (|u_2|^2) \|_{L_x^1 L_T^2}. \tag{3.30}$$

Furthermore, using Hölder's inequality, Fubini's Theorem and Sobolev embedding, we get

$$\begin{aligned}
\| v^p \partial_x v \|_{L_x^1 L_T^2} &= \| v v^{p-1} \partial_x v \|_{L_x^1 L_T^2} \\
&\leq \| v \|_{L_x^2 L_T^\infty} \| v^{p-1} \partial_x v \|_{L_x^2 L_T^2} \\
&\leq \| v \|_{L_x^2 L_T^\infty} \| v^{p-1} \|_{L_T^\infty L_x^\infty} \| \partial_x v \|_{L_T^2 L_x^2} \\
&\leq T^{1/2} \| v \|_{L_x^2 L_T^\infty} \| v^{p-1} \|_{L_T^\infty L_x^\infty} \| \partial_x v \|_{L_T^\infty L_x^2} \\
&\leq T^{1/2} \| v \|_{L_x^2 L_T^\infty} \| v \|_{L_T^\infty H_x^1}^p
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
\|\partial_x |u_j|^2\|_{L_x^1 L_T^2} &\leq \|\bar{u}_j \partial_x u_j\|_{L_x^1 L_T^2} + \|u \partial_x \bar{u}_j\|_{L_x^1 L_T^2} \\
&\leq 2\|u_j\|_{L_x^2 L_T^\infty} \|\partial_x u_j\|_{L_x^2 L_T^2} \\
&\leq 2T^{1/2} \|u_j\|_{L_x^2 L_T^\infty} \|u_j\|_{L_T^\infty H_x^1}.
\end{aligned} \tag{3.32}$$

Combining (3.30), (3.31) and (3.32), we have

$$\|\partial_x \Psi(\Delta)\|_{L_T^\infty L_x^2} \lesssim \|v_0\|_{H^1} + T^{1/2} \left(\|u_1\|_{\mathcal{S}_T}^2 + \|u_2\|_{\mathcal{S}_T}^2 + \|v\|_{\mathcal{K}_T}^{p+1} \right). \tag{3.33}$$

Now, we proceed to estimate the more delicate term, the maximal function $\|\Psi(\Delta)\|_{L_x^2 L_T^\infty}$. For this purpose we put

$$g(\Delta) = \beta v^p \partial_x v + \alpha_1 \partial_x (|u_1|^2) + \alpha_2 \partial_x (|u_2|^2).$$

From Lemma 3.3-(a) we have that

$$\begin{aligned}
(1+T)^{-\mu} \|\Psi(\Delta)\|_{L_x^2 L_T^\infty} &\lesssim \|v_0\|_{H^1} + (1+T)^{-\mu} \left\| \int_0^t V(t-t') g(\Delta) dt' \right\|_{L_x^2 L_T^\infty} \\
&= \|v_0\|_{H^1} + (1+T)^{-\mu} G(\Delta).
\end{aligned} \tag{3.34}$$

Splitting g by the projection operators and using the Proposition 3.1 and the Lemma 3.5, we get

$$\begin{aligned}
G(\Delta) &\leq \left\| \int_0^t V(t-t') P_l g(\Delta) dt' \right\|_{L_x^2 L_T^\infty} + \left\| \int_0^t V(t-t') P_h g(\Delta) dt' \right\|_{L_x^2 L_T^\infty} \\
&\lesssim \|J_x^1 P_l g(\Delta)\|_{L_T^1 L_x^2} + (1+T)^\mu \|P_h g(\Delta)\|_{L_x^1 L_T^2} \\
&\lesssim \|g(\Delta)\|_{L_T^1 L_x^2} + (1+T)^\mu \|g(\Delta)\|_{L_x^1 L_T^2}.
\end{aligned} \tag{3.35}$$

On the other hand

$$\begin{aligned}
\|g(\Delta)\|_{L_T^1 L_x^2} &\leq \|\beta v^p \partial_x v + \alpha_1 \partial_x (|u_1|^2) + \alpha_2 \partial_x (|u_2|^2)\|_{L_T^1 L_x^2} \\
&\lesssim T \left(\|v^p\|_{L_T^\infty L_x^\infty} \|\partial_x v\|_{L_T^\infty L_x^2} + \sum_{j=1}^2 \|u_j\|_{L_T^\infty L_x^\infty} \|\partial_x u_j\|_{L_T^\infty L_x^2} \right) \\
&\lesssim T \left(\|v\|_{L_T^\infty H_x^1}^{p+1} + \|u_1\|_{L_T^\infty H_x^1}^2 + \|u_2\|_{L_T^\infty H_x^1}^2 \right) \\
&\lesssim T \left(\|u_1\|_{\mathcal{S}_T}^2 + \|u_2\|_{\mathcal{S}_T}^2 + \|v\|_{\mathcal{K}_T}^{p+1} \right).
\end{aligned}$$

The same arguments as in (3.31)-(3.32) yield

$$\|g(\Delta)\|_{L_x^1 L_T^2} \leq T^{1/2} \left(\|u_1\|_{\mathcal{S}_T}^2 + \|u_2\|_{\mathcal{S}_T}^2 + \|v\|_{\mathcal{K}_T}^{p+1} \right).$$

Hence,

$$\|G(\Delta)\|_{L_x^1 L_T^2} \leq \left(T + (1+T)^\mu T^{1/2} \right) \left(\|u_1\|_{\mathcal{S}_T}^2 + \|u_2\|_{\mathcal{S}_T}^2 + \|v\|_{\mathcal{K}_T}^{p+1} \right). \tag{3.36}$$

Combining (3.34), (3.35) and (3.36), one obtains

$$(1+T)^{-\mu} \|\Psi(\Delta)\|_{L_x^2 L_T^\infty} \lesssim \|v_0\|_{H^1} + (T+T^{1/2}) \left(\|u_1\|_{\mathcal{S}_T}^2 + \|u_2\|_{\mathcal{S}_T}^2 + \|v\|_{\mathcal{K}_T}^{p+1} \right). \quad (3.37)$$

From (3.29), (3.33) and (3.37) there are positive constants \tilde{C}_0 and \tilde{C} , where \tilde{C} depends on the parameters β and α_j such that

$$\begin{aligned} \|\Psi(\Delta)\|_{\mathcal{K}_T} &\leq \tilde{C}_0 \|v_0\|_{H^1} + \tilde{C} (T+T^{1/2}) \left(\|u_1\|_{\mathcal{S}_T}^2 + \|u_2\|_{\mathcal{S}_T}^2 + \|v\|_{\mathcal{K}_T}^{p+1} \right) \\ &\leq \tilde{C}_0 \|v_0\|_{H^1} + \tilde{C} (T+T^{1/2}) \left(M_1^2 + M_2^2 + M^{p+1} \right) \end{aligned} \quad (3.38)$$

Now we put

$$M_j = 2C_0 \|u_{j0}\|_{H^1} \quad \text{and} \quad M = 2\tilde{C}_0 \|v_0\|_{H^1} \quad (3.39)$$

and choose $T > 0$ satisfying the conditions

$$\begin{cases} T \leq \frac{M_j}{2C_j} (M_j M + M_j^{q_j+1})^{-1}, & j = 1, 2, \\ T + T^{1/2} \leq \frac{M}{2\tilde{C}} (M_1^2 + M_2^2 + M^{p+1})^{-1}, \end{cases} \quad (3.40)$$

which guarantee the inclusion $(\Phi_1, \Phi_2, \Psi)(\mathcal{Z}_T) \subset \mathcal{Z}_T$.

In an analogous manner, it is not difficult to prove that the application (Φ_1, Φ_2, Ψ) is a contraction on \mathcal{Z}_T . Therefore, from contraction mapping principle we conclude that (Φ_1, Φ_2, Ψ) has a unique fixed point in \mathcal{Z}_T that is the solution to the integral equation (3.25) in the time interval $[0, T]$, with T maybe less than the one chosen in (3.40). The rest of the proof follows standard arguments, so we omit the details. \square

Proof of Theorem 2.1 (global theory). Let $1 \leq p < \frac{4}{3}$ and q_j satisfying

$$0 < q_j < \begin{cases} 4 & \text{if } \tau_j > 0, \\ \infty & \text{if } \tau_j \leq 0, \end{cases} \quad (3.41)$$

for $j = 1, 2$. For the system (1.1), as recorded in (1.5), (1.4) and (1.3), we have the following conserved quantities

$$Q(u_j)(t) = \|u_j(t)\|_{L^2}^2 = \|u_{j0}\|_{L^2}^2 =: Q_{j0}, \quad j = 1, 2, \quad (3.42)$$

$$H(\Delta)(t) = H(\Delta_0) =: H_0 \quad (3.43)$$

and

$$E(\Delta)(t) = E(\Delta_0) =: E_0, \quad (3.44)$$

where $\Delta = (u_1, u_2, v)$ and $\Delta_0 = (u_{10}, u_{20}, v_0)$.

We use these conserved quantities to obtain an *a priori* estimate in the energy space $H^1 \times H^1 \times H^1$. From (3.44), using the definition of E in (1.3), we obtain

$$E_0 = \sum_{j=1}^2 \|\partial_x u_j\|_{L_x^2}^2 + \|\partial_x v\|_{L_x^2}^2 - \tau \int v^{p+2} dx - \sum_{j=1}^2 \int \left[\alpha_j v |u_j|^2 dx + \tau_j |u_j|^{q_j+2} \right] dx. \quad (3.45)$$

Using the triangle and Cauchy-Schwartz inequalities, we get

$$\begin{aligned} y(t) &:= \sum_{j=1}^2 \|\partial_x u_j\|_{L_x^2}^2 + \|\partial_x v\|_{L_x^2}^2 \\ &= E_0 + \tau \int v^{p+2} dx + \sum_{j=1}^2 \int \alpha_j v |u_j|^2 dx + \sum_{j=1}^2 \tau_j \|u_j\|_{L_x^{q_j+2}}^{q_j+2}. \end{aligned} \quad (3.46)$$

Notice that if $\tau_j \leq 0$ for some $j = 1, 2$, then the corresponding term $\tau_j \|u_j\|_{L_x^{q_j+2}}^{q_j+2}$ is negative. In this case we need not estimate this term, because its contribution does not increase the right hand of (3.46). So, we only need to consider the condition $0 < q_j < \infty$, coming from the local theory. Hence, we just consider the case $\tau_j > 0$ for $j = 1, 2$. From the definition of $y(t)$, we have

$$\begin{aligned} y(t) &\leq |E_0| + |\tau| \|v\|_{L_x^{p+2}}^{p+2} + \sum_{j=1}^2 |\alpha_j| \|v |u_j|^2\|_{L_x^1} + \sum_{j=1}^2 \tau_j \|u_j\|_{L_x^{q_j+2}}^{q_j+2} \\ &\lesssim |E_0| + \|v\|_{L_x^{p+2}}^{p+2} + \sum_{j=1}^2 \|v\|_{L_x^2} \|u_j\|_{L_x^4}^2 + \sum_{j=1}^2 \|u_j\|_{L_x^{q_j+2}}^{q_j+2}. \end{aligned} \quad (3.47)$$

An use of Young's inequality in (3.47), yields

$$y(t) \leq C \left(|E_0| + \|v\|_{L_x^{p+2}}^{p+2} + \|v\|_{L_x^2}^2 + \sum_{j=1}^2 \|u_j\|_{L_x^4}^4 + \sum_{j=1}^2 \|u_j\|_{L_x^{q_j+2}}^{q_j+2} \right). \quad (3.48)$$

Now, we use Gagliardo-Nirenberg and Young's inequalities (taking into account that the final exponent for $\|\partial_x v\|_{L_x^2}$ cannot exceed 2), to obtain

$$\|v\|_{L_x^{p+2}}^{p+2} \lesssim \|\partial_x v\|_{L_x^2}^{\frac{p}{2}} \|v\|_{L_x^2}^{\frac{p+4}{2}} \leq \frac{1}{2C} \|\partial_x v\|_{L_x^2}^2 + C_1 \|v\|_{L_x^2}^{2\frac{4+p}{4-p}}, \quad (3.49)$$

where appears the restriction $0 < p < 4$.

Similarly, we can obtain

$$\|u_j\|_{L_x^{q_j+2}}^{q_j+2} \leq \frac{1}{16C} \|\partial_x u_j\|_{L_x^2}^2 + C_2 \|u_j\|_{L_x^2}^{2\frac{4+q_j}{4-q_j}} = \frac{1}{16C} \|\partial_x u_j\|_{L_x^2}^2 + C_2 Q_{j0}^{\frac{4+q_j}{4-q_j}}, \quad (3.50)$$

with $0 < q_j < 4$, and also

$$\begin{aligned} \|u_j\|_{L_x^4}^4 &\lesssim \|\partial_x u_j\|_{L_x^2} \|u_j\|_{L_x^2}^3 \\ &\leq \frac{1}{16C} \|\partial_x u_j\|_{L_x^2}^2 + C_3 \|u_j\|_{L_x^2}^6 \\ &= \frac{1}{16C} \|\partial_x u_j\|_{L_x^2}^2 + C_3 Q_{j0}^3, \end{aligned} \quad (3.51)$$

for $j = 1, 2$.

On the other hand, from (3.43), using definition of $H(\Delta)$ in (1.4), one gets after applying Triangle and Cauchy-Schwartz inequalities that

$$\begin{aligned} \|v\|_{L_x^2}^2 &\leq |H_0| + 2 \sum_{j=1}^2 \|u_j\|_{L_x^2} \|\partial_x u_j\|_{L_x^2} \\ &\leq |H_0| + \sum_{j=1}^2 \left[\frac{1}{16C} \|\partial_x u_j\|_{L_x^2}^2 + C_4 \|u_j\|_{L_x^2}^2 \right] \\ &= |H_0| + \sum_{j=1}^2 \left[\frac{1}{16C} \|\partial_x u_j\|_{L^2}^2 + C_4 Q_{j0} \right]. \end{aligned} \quad (3.52)$$

Inserting the first inequality in (3.52) into (3.49), one gets

$$\begin{aligned} \|v\|_{L^{p+2}}^{p+2} &\leq \frac{1}{2C} \|\partial_x v\|_{L^2}^2 + C_1 \left(|H_0| + 2 \sum_{j=1}^2 \|u_j\|_{L_x^2} \|\partial_x u_j\|_{L_x^2} \right)^{\frac{4+p}{4-p}} \\ &\leq \frac{1}{2C} \|\partial_x v\|_{L^2}^2 + C_5 |H_0|^{\frac{4+p}{4-p}} + C_5 \sum_{j=1}^2 \|u_j\|_{L^2}^{\frac{4+p}{4-p}} \|\partial_x u_j\|_{L^2}^{\frac{4+p}{4-p}} \\ &\leq \frac{1}{2C} \|\partial_x v\|_{L^2}^2 + C_5 |H_0|^{\frac{4+p}{4-p}} + \sum_{j=1}^2 \left[C_6 \|u_j\|_{L^2}^{\frac{2(4+p)}{4-3p}} + \frac{1}{16C} \|\partial_x u_j\|_{L^2}^2 \right] \\ &= \frac{1}{2C} \|\partial_x v\|_{L^2}^2 + C_5 |H_0|^{\frac{4+p}{4-p}} + \sum_{j=1}^2 \left[C_6 Q_{j0}^{\frac{4+p}{4-3p}} + \frac{1}{16C} \|\partial_x u_j\|_{L^2}^2 \right], \end{aligned} \quad (3.53)$$

where we have used Young's inequality in the third estimate with the condition $2\frac{4+p}{4-p} > 1$, and this implies $0 < p < 4/3$. This condition combined with the restriction of the local theory gives the necessary restriction $1 \leq p < 4/3$.

Now, we use (3.50), (3.51) and (3.53) in (3.48) and (3.52), to obtain

$$y(t) + \|v\|_{L^2}^2 \leq \frac{1}{2} y(t) + C_7 \left(|E_0| + |H_0| + |H_0|^{\frac{4+p}{4-p}} + \sum_{j=1}^2 \left[Q_{j0} + Q_{j0}^3 + Q_{j0}^{\frac{4+q_j}{4-q_j}} + Q_{j0}^{\frac{4+p}{4-3p}} \right] \right)$$

and consequently,

$$y(t) + \|v\|_{L^2}^2 \leq \tilde{C} \left(|E_0| + |H_0| + |H_0|^{\frac{4+p}{4-p}} + \sum_{j=1}^2 \left[Q_{j0} + Q_{j0}^3 + Q_{j0}^{\frac{4+q_j}{4-q_j}} + Q_{j0}^{\frac{4+p}{4-3p}} \right] \right) \quad (3.54)$$

From (3.54), using similar estimates to those used in the previous process and (3.42), we obtain the following *a priori* estimate:

$$\|u_1(t)\|_{H^1}^2 + \|u_2(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 \leq f(\|u_{10}\|_{H^1}, \|u_{20}\|_{H^1}, \|v_0\|_{H^1}). \quad (3.55)$$

The estimate (3.55) can be used to iterate the local existence argument to a prove the existence of the solution in any time interval $[0, T]$, for arbitrary $T > 0$. \square

3.3. Local and global theory for the (1+2)-component NLS-gKdV. In this subsection, we provide a proof of the well-posedness result for the (1+2)-component NLS-gKdV system (1.2). Here, we assume

$$q > 0 \quad \text{and} \quad p_j = \frac{n_1}{n_2} \geq 1 \quad \text{with} \quad n_1, n_2 \in \mathbb{N} \quad \text{and} \quad n_2 \text{ odd}, \quad (j = 1, 2). \quad (3.56)$$

Proof of Theorem 2.2. The proof of Theorem 2.2 is very similar to that of Theorem 2.1. In this case, to obtain the local well-posedness result in the framework of the proof of Theorem 2.1, we define

$$M = 2C_0\|u_0\|_{H^1} \quad \text{and} \quad M_j = 2\tilde{C}_0\|v_{j0}\|_{H^1}, \quad j = 1, 2 \quad (3.57)$$

and choose $T > 0$ satisfying the conditions

$$\begin{cases} T \leq \frac{M}{2C_1} (M(M_1 + M_2) + M^{q+1})^{-1}, \\ T + T^{1/2} \leq \frac{M_j}{2\tilde{C}_j} (M^2 + M_j^{p_j+1})^{-1}, \end{cases} \quad (3.58)$$

to perform the contraction mapping argument in a function space

$$\mathcal{Z}_T = \left\{ (u, v_1, v_2) \in C([0, T]; H^1 \times H^1 \times H^1); \|u\|_{\mathcal{S}_T} \leq M \text{ and } \|v_j\|_{\mathcal{K}_T} \leq M_j \right\},$$

where

$$\|u\|_{\mathcal{S}_T} := \|J_x^1 u\|_{L_T^\infty L_x^2} + (1 + T)^{-\mu} \|u\|_{L_x^2 L_T^\infty} \quad j = 1, 2, \quad (3.59)$$

$$\|v_j\|_{\mathcal{K}_T} := \|J_x^1 v_j\|_{L_T^\infty L_x^2} + (1 + T)^{-\mu} \|v_j\|_{L_x^2 L_T^\infty}, \quad (3.60)$$

and the norm on \mathcal{Z}_T is defined by

$$\|(u, v_1, v_2)\|_{\mathcal{Z}_T} := \|u\|_{\mathcal{S}_T} + \|v_1\|_{\mathcal{K}_T} + \|v_2\|_{\mathcal{K}_T}. \quad (3.61)$$

The proof of the global well-posedness result is also similar to that of Theorem 2.1. Considering, $0 < q < 4$ and $1 \leq p_j < \frac{4}{3}$, $j = 1, 2$, one can use the conserved quantities stated in (1.6), (1.7) and (1.8), to get an *a priori* estimate

$$\|u(t)\|_{H^1}^2 + \|v_1(t)\|_{H^1}^2 + \|v_2(t)\|_{H^1}^2 \leq f(\|u_0\|_{H^1}, \|v_{10}\|_{H^1}, \|v_{20}\|_{H^1}), \quad (3.62)$$

which can be used to extend the local solution to any arbitrary time interval $[0, T]$, $T > 0$. So, we omit the details. \square

3.4. Local theory below the energy space. The proof of the local well-posedness results stated in Theorems 2.3 and 2.4 follow with an standard contraction mapping argument in the Bourgain's space framework. The main ingredients are the linear, bilinear and trilinear estimates recorded in Lemmas 3.7–3.11.

In fact, to prove Theorem 2.3, we define

$$\begin{cases} \Phi_j(\Delta) = \psi(t)S(t)u_{j0} + i\psi_T(t) \int_0^t S(t-t') [\alpha_1(u_j v) + \gamma_j |u_j|^2 u_j] (t') dt' \\ \Psi(\Delta) = \psi(t)V(t)v_0 - \psi_T(t) \int_0^t V(t-t') \partial_x [\beta v^2 + \alpha_1 |u_1|^2 + \alpha_2 |u_2|^2] (t') dt'. \end{cases} \quad (3.63)$$

Taking $M_j = 2C_0\|u_{j0}\|_{H^s}$, $j = 1, 2$, $M = 2C_0\|v_0\|_{H^k}$, and for $0 < \epsilon \ll 1$

$$T^\epsilon \leq \frac{1}{2} \min \left\{ \frac{1}{C_1[M + M_1^2]}, \frac{1}{C_2[M + M_2^2]}, \frac{M}{C[M^2 + M_1^2 + M_2^2]} \right\}, \quad (3.64)$$

one can show that (Φ_1, Φ_2, Ψ) is a contraction mapping on

$$\mathcal{Z}^\epsilon := \left\{ (u_1, u_2, v) \in X^{s, \frac{1}{2}+\epsilon} \times X^{s, \frac{1}{2}+\epsilon} \times Y^{k, \frac{1}{2}+\epsilon} : \|u_j\|_{X^{s, \frac{1}{2}+\epsilon}} \leq M_j, \|v\|_{Y^{k, \frac{1}{2}+\epsilon}} \leq M \right\},$$

with a norm defined by

$$\|(u_1, u_2, v)\|_{\mathcal{Z}^\epsilon} := \|u_1\|_{X^{s, \frac{1}{2}+\epsilon}} + \|u_2\|_{X^{s, \frac{1}{2}+\epsilon}} + \|v\|_{Y^{k, \frac{1}{2}+\epsilon}}.$$

This process is now a classical one, so we omit the details.

The proof of Theorem 2.4 follows analogously.

At this point, a natural question arises about global well-posedness below energy spaces. For this, the recently introduced *I-method* [12, 13] combined with almost conserved quantities could be useful. This will be addressed elsewhere.

4. Existence of Prescribed L^2 -Norm Solutions

This section is devoted to the proof of the existence of nontrivial normalized solutions of the system (1.9). Throughout this section, we assume that all conditions of (2.11) hold and that $0 < p < 4$.

4.1. The variational problem. We study the following problem: for given $(r, l, m) \in \mathbb{R}_+^2 \times \mathbb{R}$, find a function $(\phi_1, \phi_2, w) \in S_r \times S_l \times K_m$ such that $E(\phi_1, \phi_2, w) = \Theta(r, l, m)$, where $\Theta(r, l, m)$ is defined by

$$\Theta(r, l, m) = \inf \{ E(f_1, f_2, g) : (f_1, f_2, g) \in S_r \times S_l \times K_m \}. \quad (4.1)$$

Before we proceed, let us fix some notations that will be used in the sequel:

$$\Sigma_{r,l,m} = S_{r \times l} \times K_m, \text{ where } S_{r \times l} = S_r \times S_l,$$

$$E_1(f) = E(f, 0, 0), \quad E_2(g) = E(0, g, 0), \quad E_3(h) = E(0, 0, h),$$

$$E_{12}(f, g) = E(f, g, 0), \quad E_{23}(g, h) = E(0, g, h), \quad E_{13}(f, h) = E(f, 0, h),$$

and

$$F_j(f_j, h) = \alpha_j \int_{-\infty}^{\infty} |f_j|^2 h \, dx, \quad j = 1, 2.$$

Lemma 4.1. *The function $\Theta(r, l, m)$ is finite and negative.*

Proof. Take an element $\Delta = (f_1, f_2, g)$ of $\Sigma_{r,l,m}$. Using the Gagliardo-Nirenberg inequality and the Cauchy-Schwarz inequality, it follows that there exists a constant $C = C(q_1, \epsilon)$ such that

$$\begin{aligned} \|f_1\|_{L^{q_1+2}}^{q_1+2} &\leq C \|\partial_x f_1\|_{L^2}^{q_1/2} \|f_1\|_{L^2}^{(q_1/2)+2} \\ &\leq \epsilon \|\partial_x f_1\|_{L^2}^2 + C \|f_1\|_{L^2}^{2+4q_1/(4-q_1)} = \epsilon \|\partial_x f_1\|_{L^2}^2 + C r^{1+2q_1/(4-q_1)}, \end{aligned} \quad (4.2)$$

where $\epsilon > 0$ can be chosen arbitrarily small. Using Hölder's inequality, we obtain that

$$\int_{-\infty}^{\infty} |f_1|^2 |g| \, dx \leq C \|f_1\|_{L^4}^2 \|g\|_{L^2} \leq \epsilon \|\partial_x f_1\|_{L^2}^2 + Cr(1+t). \quad (4.3)$$

Similar estimates hold for $\|f_2\|_{L^{q_2+2}}^{q_2+2}$, $\|g\|_{L^{p+2}}^{p+2}$, and $\int_{-\infty}^{\infty} |f_2|^2 |g| \, dx$. These estimates will be used repeatedly throughout the rest of the paper. With the aid of the estimates (4.2) and (4.3), one can infer that

$$E(\Delta) \geq (1-\epsilon) \sum_{j=1}^2 \|f_j\|_{H^1}^2 + (1-\epsilon) \|g\|_{H^1}^2 - C_{\epsilon,r,l,m} - (r+l+m).$$

Taking $\epsilon < 1$, we now obtain

$$E(\Delta) \geq -C_{\epsilon,r,l,m} - (r+l+m) > -\infty.$$

To prove $\Theta(r, l, m) < 0$, take any $\Delta = (f_1, f_2, g) \in \Sigma_{r,l,m}$ such that $f_1(x) > 0$, $f_2(x) > 0$, and $g(x) > 0$ for all $x \in \mathbb{R}$. For arbitrary $\theta > 0$, define the scaling functions

$$f_{j\theta}(\cdot) = \theta^{1/2} f_j(\cdot\theta), \quad j = 1, 2, \quad \text{and} \quad g_\theta(\cdot) = \theta^{1/2} g(\cdot\theta).$$

It is easy to see that $\Delta_\theta = (f_{1\theta}, f_{2\theta}, g_\theta)$ belongs to $\Sigma_{r,l,m}$ as well and hence, one has

$$E(\Delta_\theta) \leq \theta^2 \int_{-\infty}^{\infty} (|\partial_x f_1|^2 + |\partial_x f_2|^2 + |\partial_x g|^2) \, dx - \theta^{1/2} \sum_{j=1}^2 \alpha_j |f_j|^2 g \, dx.$$

Upon taking θ small enough, it is obvious that $E(\Delta_\theta) < 0$ and hence, the infimum $\Theta(r, l, m)$ defined in (4.1) is negative. \square

In what follows we call a sequence $\{(f_{1n}, f_{2n}, g_n)\}$ of functions in \mathcal{H} an (r, l, m) -admissible if the following conditions hold:

$$\lim_{n \rightarrow \infty} Q(f_{1n}) = r, \quad \lim_{n \rightarrow \infty} Q(f_{2n}) = l, \quad \text{and} \quad \lim_{n \rightarrow \infty} Q(g_n) = m.$$

We will say that an (r, l, m) -admissible sequence $\{(f_{1n}, f_{2n}, g_n)\}$ of functions in \mathcal{H} is a minimizing sequence for $\Theta(r, l, m)$ if it satisfies the condition

$$\lim_{n \rightarrow \infty} E(f_{1n}, f_{2n}, g_n) = \Theta(r, l, m). \quad (4.4)$$

Using the estimates obtained in Lemma 4.1, it is easy to prove that such a sequence is bounded. The common element in the proof of the relative compactness of minimizing sequence via concentration compactness argument is to show the strict subadditivity condition of the problem. In the present situation, however, this is considerably difficult by the fact that the function $\Theta(r, l, m)$ consists of three independent parameters. To overcome this difficulty we utilize the properties of symmetric rearrangement of functions to carry a careful analysis of minimizing sequences. In the next few lemmas we will be devoted to proving the strict sub-additivity of $\Theta(r, l, m)$.

The first lemma establishes some special properties of minimizing sequences.

Lemma 4.2. *Suppose $\{(f_{1n}, f_{2n}, g_n)\}$ be an (r, l, m) -admissible sequence satisfying the condition (4.4). Then the following properties hold:*

(i) *if $s > 0$ and $r, l \geq 0$, then there exists a pair of numbers $(\delta_1, N_1) \in (0, \infty) \times \mathbb{N}$ such that $\|\partial_x g_n\|_{L^2} \geq \delta_1$ holds for all $n \geq N_1$.*

(ii) *if $r > 0$ and $l, m \geq 0$, then there exists a pair $(\delta_2, N_2) \in (0, \infty) \times \mathbb{N}$ such that $\|\partial_x f_{1n}\|_{L^2} \geq \delta_2$ for all $n \geq N_2$. A similar assertion holds for $\|\partial_x f_{2n}\|_{L^2}$ when r and l are switched.*

We prove each part separately.

Proof of Lemma 4.2 (i). Suppose, for the sake of contradiction, that the conclusion of part (i) is not true. Then, passing to an appropriate subsequence if necessary, we may assume that $\lim_{n \rightarrow \infty} \|\partial_x g_n\|_{L^2} = 0$. Using the estimates obtained in Lemma 4.1, one sees that

$$\int_{-\infty}^{\infty} |g_n|^{p+2} dx \rightarrow 0 \quad \text{and} \quad \int_{-\infty}^{\infty} |f_{jn}|^2 |g_n| dx \rightarrow 0, \quad j = 1, 2,$$

as $n \rightarrow \infty$. In consequence, we come to the identity

$$\Theta(r, l, m) = \lim_{n \rightarrow \infty} E(f_{1n}, f_{2n}, g_n) = \lim_{n \rightarrow \infty} \sum_{j=1}^2 E_j(f_{jn}). \quad (4.5)$$

Now, take any $w \in S_m$ such that $w \geq 0$. For an arbitrary $\theta > 0$, we set $w_\theta(\cdot) = \theta^{1/2} w(\cdot/\theta)$. Then the scaling function w_θ also belongs to S_m and hence, for all n , there obtains

$$\Theta(r, l, m) \leq E(f_{1n}, f_{2n}, w_\theta). \quad (4.6)$$

On the other hand, for small enough θ , it is obvious that

$$A_w := \theta^2 \int_{-\infty}^{\infty} |\partial_x w|^2 dx - \tau \theta^{p/2} \int_{-\infty}^{\infty} w^{p+2} dx < 0, \quad (4.7)$$

Using this notation, one obtains from the inequality (4.6) that

$$\Theta(r, l, m) \leq \sum_{j=1}^2 (E_j(f_{jn}) - \theta^{1/2} F_j(f_{jn}, w)) + A_w \leq \sum_{j=1}^2 E_j(f_{jn}) + A_w.$$

Upon passing limit as $n \rightarrow \infty$ to the last inequality, we obtain that

$$\Theta(r, l, m) \leq \lim_{n \rightarrow \infty} \sum_{j=1}^2 E_j(f_{jn}) + A_w,$$

but this last inequality contradicts (4.5) and (4.7), and hence, the result (i) follows. \square

To prove part (ii), we shall make use of the following result of [2] concerning the existence of solutions of a certain problem closely related to (4.1).

Lemma 4.3. *For $(\phi, w) \in H_{\mathbb{C}}^1 \times H_{\mathbb{R}}^1$, define $E_{23}(\phi, w) = E(0, \phi, w)$. Then, for every $l > 0$ and $m > 0$, there exists a solution (ϕ_0, w_m) to the problem*

$$\nu(l, m) = \inf \{E_{23}(\phi, w) : \phi \in S_l \text{ and } w \in K_m\}.$$

Furthermore, $w_m(x) > 0$ for all $x \in \mathbb{R}$ and there exists a positive \mathbb{R} -valued function ϕ_l such that $\phi_0 = \phi_l$ up to a phase factor. In particular, $E_{23}(\phi_l, w_m) = \nu(l, m)$.

The next two lemmas are well-known uniqueness results (for details and further discussion, we refer readers to [9]).

Lemma 4.4. *Suppose $W_p \in H_{\mathbb{R}}^1$ is a non-zero solution of*

$$-Q'' + \lambda_3 Q = \frac{\beta}{p+1} Q^{p+1},$$

where $\lambda_3 \in \mathbb{R}$. Then $\lambda_3 > 0$ and $W_p(x) = w(x + x_0)$, where $x_0 \in \mathbb{R}$ and $w_p(x)$ has the following explicit expression

$$w_p(x) = \left(\frac{(p+1)(p+2)\lambda_3}{2\beta} \right)^{1/p} \text{sech}^{2/p} \left(\frac{\sqrt{\lambda_3} px}{2} \right). \quad (4.8)$$

The second lemma concerns about the uniqueness of solutions of the equations,

$$\begin{cases} -Q_j'' + \lambda_j Q_j = (q_j + 2)\tau_j |Q_j|^{q_j+1}, \\ Q_j \in H_{\mathbb{C}}^1 \setminus \{0\}, \lambda_j \in \mathbb{R}, j = 1, 2. \end{cases} \quad (4.9)$$

Lemma 4.5. *There is, up to translation and phase shift, a unique solution of the equation (4.9), i.e., the set of all solutions of (4.9) is of the form*

$$\mathcal{G}_j = \{ e^{i\theta_j} \psi_{q_j}(\cdot + x_0) : \theta_j, x_0 \in \mathbb{R} \},$$

where $\psi_{q_j}(x)$ is explicitly given by

$$\psi_{q_j}(x) = \left(\frac{\lambda_j}{2\tau_j} \right)^{1/q_j} \text{sech}^{2/q_j} \left(\frac{\sqrt{\lambda_j} q_j x}{2} \right). \quad (4.10)$$

We are now able to prove part (ii) of Lemma 4.2.

Proof of Lemma 4.2 (ii). The proof is again carried out by contradiction. As before, by extracting a subsequence if necessary, one assumes that $\lim_{n \rightarrow \infty} \|\partial_x f_{1n}\|_{L^2} = 0$. Then, using the estimates (4.2) and (4.3) yet again, one easily verifies that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f_{1n}|^2 |g_n| dx = 0 = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f_{1n}|^{q_1+2} dx,$$

and as a result, we come to the identity

$$\Theta(r, l, m) = \lim_{n \rightarrow \infty} [E_2(f_{2n}) + E_3(g_n) - F_2(f_{2n}, g_n)]. \quad (4.11)$$

The following four cases are possible: (i) $l > 0$ and $m > 0$; (ii) $l > 0$ and $m = 0$; (iii) $l = 0$ and $m > 0$; and (iv) $l = 0$ and $m = 0$. If $(l, m) \in \mathbb{R}_+^2$, let the functions ϕ_l and w_m be as defined in Lemma 4.3. Then $\Theta(r, l, m) \geq E_{23}(\phi_l, w_m)$. To arrive at a contradiction, it is claimed that there exists $R_r \in S_r$ such that

$$\|\partial_x R_r\|_{L^2}^2 - F_1(R_r, w_m) < 0. \quad (4.12)$$

To prove (4.12), take any $\rho \in C_c^\infty$ such that $\rho \geq 0$, $\rho(0) = 1$, and $\|\rho\|_{L^2}^2 = r$. For an arbitrary $\theta > 0$, we set $\rho_\theta(x) = \theta^{1/2}\rho(\theta x)$. Then $\rho_\theta \in S_r$. Since the function w_m is integrable over \mathbb{R} , an application of Lebesgue dominated convergence theorem yields

$$\|\partial_x \rho_\theta\|_{L^2}^2 - F_1(\rho_\theta, w_m) \leq \theta^2 \int_{-\infty}^{\infty} |\partial_x \rho|^2 dx - \theta \int_{-\infty}^{\infty} w_m(x) dx.$$

Since $\int_{-\infty}^{\infty} w_m(x) dx > 0$, it follows that $\|\partial_x \rho_\theta\|_{L^2}^2 - F_1(\rho_\theta, w_m) < 0$ for sufficiently small θ . Thus $R_r = \rho_\theta$ satisfies (4.12) whenever θ is sufficiently small. With (4.12) in hand, we now obtain

$$\begin{aligned} \Theta(r, l, m) &\leq E(R_r, \phi_l, w_m) \\ &= E_{23}(\phi_l, w_m) + (\|\partial_x R_r\|_{L^2}^2 - F_1(R_r, w_m)) - \tau_1 \int_{-\infty}^{\infty} |R_r|^{q_1+2} dx \\ &\leq E_{23}(\phi_l, w_m) + (\|\partial_x R_r\|_{L^2}^2 - F_1(R_r, w_m)) < E_{23}(\phi_l, w_m), \end{aligned}$$

which is a contradiction. Next, consider the case that $l > 0$ and $m = 0$. The uniqueness result (Lemma 4.5) implies that for any $l > 0$, any solution of the problem $\inf\{E_2(g) : g \in S_l\}$ is of the form $\psi_l = e^{i\theta_0} T_{x_0} \psi_{q_2}$, where $\theta_0, x_0 \in \mathbb{R}$ and ψ_{q_2} is as defined in Lemma 4.5. Then, from (4.11), we have that $\Theta(r, l, m) \geq E_2(\psi_l)$. To arrive at a contradiction, take any $(f, w) \in S_r \times K_m$ such that $f(x) > 0$ and $w(x) > 0$ for $x \in \mathbb{R}$. For an arbitrary $\theta > 0$, define $f_\theta(x) = \theta^{1/2}f(\theta x)$ and $w_\theta(x) = \theta^{1/2}w(\theta x)$. Then it is obvious that $(f_\theta, w_\theta) \in S_r \times K_m$ and hence,

$$\begin{aligned} E_{13}(f_\theta, w_\theta) &= E_1(f_\theta) + E_3(w_\theta) - F_1(f_\theta, w_\theta) \\ &\leq \theta^2 \int_{-\infty}^{\infty} (|\partial_x f|^2 + |\partial_x w|^2) dx - \theta^{1/2} \int_{-\infty}^{\infty} \alpha_1 |f|^2 w dx, \end{aligned}$$

from which it is concluded that $E_{13}(f_\theta, w_\theta) < 0$ whenever θ is sufficiently small. In consequence of the preceding inequality, one has that

$$\begin{aligned} \Theta(r, l, m) &\leq E(f_\theta, \psi_l, w_\theta) = E_2(\psi_l) + E_{13}(f_\theta, w_\theta) - F_2(\psi_l, w_\theta) \\ &\leq E_2(\psi_l) + E_{13}(f_\theta, w_\theta) < E_2(\psi_l), \end{aligned}$$

a contradiction. The case that $l = 0$ and $m > 0$ is similar. Finally, suppose that $l = 0$ and $m = 0$. Then, from (4.11), one has $\Theta(r, l, m) \geq 0$. On the other hand,

$$\Theta(r, l, m) = \inf \{E_1(f) : f \in S_r\},$$

and we can make $E_1(f) < 0$ by taking the scaling function $f_\theta(x) = \theta^{1/2}f(\theta x)$ defined as before. This in turn implies that $\Theta(r, l, m) < 0$, a contradiction. This completes the proof in all cases. The conclusion for $\partial_x f_{2n}$ can be proved by using an analogous argument. \square

Another important ingredient for the proof of strict sub-additivity is the following lemma, which concerns the existence of special minimizing sequences for $\Theta(r, l, m)$.

Lemma 4.6. *There exist a minimizing sequence $\{(f_{1n}, f_{2n}, g_n)\}$ for $\Theta(r, l, m)$ such that for each n , the functions f_{1n} , f_{2n} , and g_n are \mathbb{R} -valued, non-negative, C_c^∞ , even,*

non-increasing on the set $[0, \infty)$, and satisfy the condition

$$\|f_{1n}\|_{L^2}^2 = r, \quad \|f_{2n}\|_{L^2}^2 = l, \quad \text{and} \quad \|g_n\|_{L^2}^2 = m.$$

Proof. Start with a given minimizing sequence (p_{1n}, p_{2n}, q_n) for $\Theta(r, l, m)$. Without loss of generality we may assume that $(r, l, m) \in \mathbb{R}_+^3$. First approximate (p_{1n}, p_{2n}, q_n) by compactly supported functions (c_{1n}, c_{2n}, d_n) . For a non-negative measurable function f , let f^* denotes its symmetric decreasing rearrangement (for details, see Chapter 3 of [24]). Using rearrangement inequalities (cf. Chapter 7 of [24]), we have that

$$E(|f_1|^*, |f_2|^*, |g|^*) \leq E(f_1, f_2, g)$$

for any $(f_1, f_2, g) \in \mathcal{H}$. Hence, one can assume without loss of generality that $c_{1n} = |c_{1n}|^*$, $c_{2n} = |c_{2n}|^*$, and $d_n = |d_n|^*$ hold. Now let $\psi \in C_c^\infty$ be any non-negative, even, and decreasing function on the set $[0, \infty)$, which also satisfies the condition $\int_{-\infty}^{\infty} \psi(x) dx = 1$. For any arbitrary $\epsilon > 0$, consider $\psi_\epsilon(\cdot) = (1/\epsilon)\psi(\cdot/\epsilon)$, and set

$$f_{1n} = \frac{r^{1/2} (c_{1n} \star \psi_{\epsilon_n})}{\|c_{1n} \star \psi_{\epsilon_n}\|_{L^2}}, \quad f_{2n} = \frac{s^{1/2} (c_{2n} \star \psi_{\epsilon_n})}{\|c_{2n} \star \psi_{\epsilon_n}\|_{L^2}}, \quad g_n = \frac{t^{1/2} (d_n \star \psi_{\epsilon_n})}{\|d_n \star \psi_{\epsilon_n}\|_{L^2}},$$

with ϵ_n chosen approximately small whenever n is large. Since ψ_{ϵ_n} is a mollifier, it follows that the sequence $\{(f_{1n}, f_{2n}, g_n)\}$ is the desired minimizing sequence. \square

With the properties of minimizing sequences given in Lemmas 4.2 and 4.6 in hand, we are now able to prove the strict subadditivity inequality for the function $\Theta(r, l, m)$. This will be an essential ingredient later in ruling out the case of dichotomy.

Lemma 4.7. *The function $\Theta(r, l, m)$ enjoys the following strict subadditivity property*

$$\Theta(r_1 + r_2, l_1 + l_2, m_1 + m_2) < \sum_{i=1}^2 \Theta(r_i, l_i, m_i), \quad r_i, l_i, m_i \geq 0. \quad (4.13)$$

Proof. We may assume that $r_1 + r_2 > 0$, $l_1 + l_2 > 0$, $m_1 + m_2 > 0$, $r_1 + l_1 + m_1 > 0$, and $r_2 + l_2 + m_2 > 0$; otherwise (4.13) reduces to the strict subadditivity inequality of the function with fewer parameters. For $i = 1, 2$, consider the special minimizing sequences $\{(f_{1n}^{(i)}, f_{2n}^{(i)}, g_n^{(i)})\}$ for $\Theta(r_i, l_i, m_i)$, as constructed in Lemma 4.6. For each n , select the numbers x_n such that

$$\text{supp } f_{jn}^{(1)} \cap \text{supp } T_{x_n} f_{jn}^{(2)} = \emptyset \quad \text{and} \quad \text{supp } g_n^{(1)} \cap \text{supp } T_{x_n} g_n^{(2)} = \emptyset,$$

and define the sequence $\{(f_{1n}, f_{2n}, g_n)\}$ of functions by setting

$$(f_{1n}, f_{2n}, g_n) = \left(\left(f_{1n}^{(1)} + T_{x_n} f_{1n}^{(2)} \right)^*, \left(f_{2n}^{(1)} + T_{x_n} f_{2n}^{(2)} \right)^*, \left(g_n^{(1)} + T_{x_n} g_n^{(2)} \right)^* \right). \quad (4.14)$$

By the definition of the infimum $\Theta(r_1 + r_2, l_1 + l_2, m_1 + m_2)$, it is clear that

$$\Theta(r_1 + r_2, l_1 + l_2, m_1 + m_2) \leq E(f_{1n}, f_{2n}, g_n). \quad (4.15)$$

A lemma about symmetric rearrangement (Lemma 2.10 of [2]) now comes to our aid. The lemma states that if $f, g : \mathbb{R} \rightarrow [0, \infty)$ are non-increasing, even, C_c^∞ functions; the real numbers x_1, x_2 be such that the translated functions $T_{x_1} f$ and $T_{x_2} g$ have disjoint

supports; and $S = T_{x_1}f + T_{x_2}g$, then the first derivative $(S^*)'$ of S^* (in the distributional sense) is in L^2 and one has the estimate

$$\|(S^*)'\|_{L^2}^2 \leq \|S'\|_{L^2}^2 - \frac{3}{4} \min\{\|f'\|_{L^2}^2, \|g'\|_{L^2}^2\}. \quad (4.16)$$

Applying the estimate (4.16) to each component of the sequence (4.14) and using the well-known fact that $\|u\|_{L^p} = \|u^*\|_{L^p}$, $1 \leq p \leq \infty$, it immediately follows that

$$\begin{aligned} \|\partial_x g_n\|_{L^2}^2 + \sum_{j=1}^2 \|\partial_x f_{jn}\|_{L^2}^2 &\leq \|\partial_x g_n^{(1)}\|_{L^2}^2 + \|\partial_x(T_{x_n}g_n^{(2)})\|_{L^2}^2 \\ &+ \sum_{j=1}^2 \left(\|\partial_x f_{jn}^{(1)}\|_{L^2}^2 + \|\partial_x(T_{x_n}f_{jn}^{(2)})\|_{L^2}^2 \right) \\ &- \frac{3}{4} \left(\min\{\|\partial_x g_n^{(1)}\|_{L^2}^2, \|\partial_x g_n^{(2)}\|_{L^2}^2\} + \sum_{j=1}^2 \min\{\|\partial_x f_{jn}^{(1)}\|_{L^2}^2, \|\partial_x f_{jn}^{(2)}\|_{L^2}^2\} \right). \end{aligned} \quad (4.17)$$

Now estimating the right side of (4.15) using (4.17) and the rearrangement inequality (Chapter 3 of [24]), and passing to the limit as $n \rightarrow \infty$ in the resultant inequality, there obtains

$$\Theta(r_1 + r_2, l_1 + l_2, m_1 + m_2) \leq \sum_{i=1}^2 \Theta(r_i, l_i, m_i) - \liminf_{n \rightarrow \infty} J_n, \quad (4.18)$$

where the quantity J_n denotes the last term in (4.17) involving minimums. We now prove the strict inequality (4.13). As noted in [8], it is sufficient to consider the following cases:

- (i) $r_1, r_2 > 0$ and $l_1, l_2, m_1, m_2 \geq 0$;
- (ii) $r_1 = 0, r_2 > 0, l_2 > 0$, and $m_1 = 0$;
- (iii) $r_1 = 0, r_2 > 0, l_2 > 0$, and $m_1 > 0$;
- (iv) $r_1 = 0, r_2 > 0, l_2 = 0$, and $m_1 = 0$; and
- (v) $r_1 = 0, r_2 > 0, l_2 = 0$, and $m_1 > 0$.

All other cases can be reduced to one of these cases by switching the roles of the parameters. Consider first the case that $r_1, r_2 > 0$. Using Lemma 4.2(ii), there exist a pair of positive numbers $\{\delta_1, \delta_2\}$ such that for all large enough n , we have $\|\partial_x f_{1n}^{(1)}\|_{L^2} \geq \delta_1$ and $\|\partial_x f_{1n}^{(2)}\|_{L^2} \geq \delta_2$. Let $\delta = \min(\delta_1, \delta_2) > 0$. Then it follows that $J_n \geq 3\delta/4$ for all large enough n , and in view of (4.18), we conclude that

$$\begin{aligned} \Theta(r_1 + r_2, l_1 + l_2, m_1 + m_2) &\leq \Theta(r_1, l_1, m_1) + \Theta(r_2, l_2, m_2) - 3\delta/4 \\ &< \Theta(r_1, l_1, m_1) + \Theta(r_2, l_2, m_2). \end{aligned}$$

Next suppose the case that $r_1 = 0, r_2 > 0, l_2 > 0$, and $m_1 = 0$. Since $r_1 + l_1 + m_1 > 0$, so $l_1 > 0$ as well. Then another application of Lemma 4.2(ii) guarantees the existence of numbers $\delta_3, \delta_4 > 0$ such that for all large enough n , $\|\partial_x f_{2n}^{(1)}\|_{L^2} \geq \delta_3$ and $\|\partial_x f_{2n}^{(2)}\|_{L^2} \geq$

δ_4 . As before, set $\delta = \min(\delta_3, \delta_4) > 0$. Then it is obvious that $J_n \geq 3\delta/4$ for all large enough n , and (4.13) follows from (4.18). This completes the proof in case (ii).

We now turn to the case (iii), i.e., when $r_1 = 0, r_2 > 0, l_2 > 0$, and $m_1 > 0$. If $l_1 > 0$ or $m_2 > 0$, then the proofs follow same lines as that in the case (ii) above. Thus, we assume $l_1 = 0$ and $m_2 = 0$, and prove that

$$\Theta(r_2, l_2, m_1) < \Theta(0, 0, m_1) + \Theta(r_2, l_2, 0). \quad (4.19)$$

Take the function w_p as defined in Lemma 4.4 with $\lambda_3 > 0$ so chosen such that $w_p \in K_{m_1}$. Then the function $w_{m_1} = w_p$ satisfies the identity

$$E_3(w_p) = \inf\{E_3(h) : h \in K_{m_1}\}$$

(see, for example [9]). Similarly, let ψ_{q_1}, ψ_{q_2} be the functions as defined in Lemma 4.5 with λ_1 and λ_2 so chosen such that $(\psi_{q_1}, \psi_{q_2}) \in S_{r_2 \times l_2}$. Then the functions $\phi_{r_2} = \psi_{q_1}$ and $\phi_{l_2} = \psi_{q_2}$ satisfy the identities

$$E_1(\phi_{r_2}) = \inf\{E_1(f) : f \in S_{r_2}\} \text{ and } E_2(\phi_{l_2}) = \inf\{E_2(g) : g \in S_{l_2}\}.$$

Now the function $(\phi_{r_2}, \phi_{l_2}, w_{m_1})$ belongs to $S_{r_2 \times l_2} \times K_{m_1}$ and we come to the inequality

$$\begin{aligned} \Theta(r_2, l_2, m_1) &\leq E(\phi_{r_2}, \phi_{l_2}, w_{m_1}) = \int_{-\infty}^{\infty} (|\partial_x(w_{m_1})|^2 - \tau w_{m_1}^{p+2}) dx \\ &+ \int_{-\infty}^{\infty} (|\partial_x(\phi_{r_2})|^2 + |\partial_x(\phi_{l_2})|^2 - \tau_1 |\phi_{r_2}|^{q_1+2} - \tau_2 |\phi_{l_2}|^{q_2+2}) dx \\ &- \alpha_1 \int_{-\infty}^{\infty} |\phi_{r_2}|^2 w_{m_1} dx - \alpha_2 \int_{-\infty}^{\infty} |\phi_{l_2}|^2 w_{m_1} dx, \end{aligned} \quad (4.20)$$

It is obvious that $\int_{-\infty}^{\infty} |\phi_{r_2}|^2 w_{m_1} dx > 0$ and $\int_{-\infty}^{\infty} |\phi_{l_2}|^2 w_{m_1} dx > 0$. Then, the strict inequality (4.19) follows from (4.20). In the case (iv), one has to prove that

$$\Theta(r_2, l_1, m_2) < \Theta(0, l_1, 0) + \Theta(r_2, 0, m_2), \quad (4.21)$$

which can be done using an analogous argument as in the proof of (4.19). It only remains to prove (4.13) in case (v). Consider the case (v), i.e., $r_1 = 0, r_2 > 0, l_2 = 0$, and $m_1 > 0$. If $t_2 > 0$, then (4.13) follows by the use of part (i) of Lemma 4.2 in the inequality (4.18). Thus, one can assume $t_2 = 0$ and prove that

$$\Theta(r_2, l_1, m_1) < \Theta(0, l_1, m_1) + \Theta(r_2, 0, 0). \quad (4.22)$$

The proof of the inequality (4.22) follows along the same lines as that of (4.19) as well. This completes the proof of (4.13) in all cases. \square

We now proceed to prove the existence result for normalized solutions.

4.2. Existence result for (2+1)-component NLS-gKdV. To any minimizing sequence $\{(f_{1n}, f_{2n}, g_n)\}$ is associated, up to taking a subsequence, a number μ given by

$$\mu = \lim_{\zeta \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-\zeta}^{y+\zeta} \rho_n(x) dx, \quad (4.23)$$

where the function $\rho_n(x)$ is defined by

$$\rho_n(x) := |f_{1n}(x)|^2 + |f_{2n}(x)|^2 + g_n^2(x).$$

Then the number μ satisfies $0 \leq \mu \leq r + l + m$. We will examine separately the three (mutually exclusive) cases, $\mu = r + l + m$ (tightness), $0 < \mu < r + l + m$ (dichotomy), and $\mu = 0$ (vanishing). Once we prove the tightness $\mu = r + l + m$, then one can follow the same lines as in the proof of the fundamental Lemma I.1(i) of [23] to prove that the translated sequence $\{(T_{y_n} f_{1n}, T_{y_n} f_{2n}, T_{y_n} g_n)\}$ has a subsequence which converges in \mathcal{H} norm to a function in $\mathcal{O}_{r,l,m}$. The proof differs only in minor details and will not be repeated here. Thus, in order to prove Theorem 2.5, it suffices to rule out dichotomy and vanishing cases.

Lemma 4.8. *Suppose $(r, l, m) \in \mathbb{R}_+^3$ and let $\{(f_{1n}, f_{2n}, g_n)\}$ be an (r, l, m) -admissible sequence satisfying (4.4). Let μ be as defined in (4.23). Then there exists $(r_1, l_1, m_1) \in [0, r] \times [0, l] \times [0, m]$ satisfying $\mu = r_1 + l_1 + m_1$ and*

$$\Theta(r_1, l_1, m_1) + \Theta(r - r_1, l - l_1, m - m_1) \leq \Theta(r, l, m). \quad (4.24)$$

Proof. The proof follows along the same lines as that of Theorem 3.4 of [8], which is a generalization of Theorem 3.10 of [1]. We omit the details. \square

We are now able to prove the existence theorem.

Proof of Theorem 2.5 (existence of prescribed L^2 -norm solutions). Suppose $(r, l, m) \in \mathbb{R}_+^3$ and let $\{(f_{1n}, f_{2n}, g_n)\}$ be any minimizing sequence for $\Theta(r, l, m)$. As noted before, the existence of minimizers follows if we show that $\mu = r + l + m$, where μ is as defined in (4.23). Hence, to complete the proof of Theorem 2.5, we only have to show that

- (i) $\mu \neq 0$, and
- (ii) $\mu \notin (0, r + l + m)$.

In view of Lemmas 4.7 and 4.8, (ii) is clear. So we only have to prove that $\mu \neq 0$. A standard fact in the application of concentration compactness method (Lemma I.1 of [23]) states that if $\{u_n\}$ is bounded in L^α , $\{u'_n\}$ is bounded in L^p , and for some $\omega > 0$,

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-\omega}^{y+\omega} |u_n|^\alpha dx = 0, \quad (4.25)$$

then for every $q > \alpha$,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |u_n|^q dx = 0. \quad (4.26)$$

If $\mu = 0$, then (4.25) holds for $u_n = |f_{1n}|$, $u_n = |f_{2n}|$, and $u_n = g_n$, and for every $\alpha > 2$, f_{1n} , f_{2n} , and g_n all converge to 0 in L^α . But then, since

$$|F_j(f_{jn}, g_n)| \leq \|f_{jn}\|_{L^4}^{1/2} \|g_n\|_{L^2}, \quad j = 1, 2,$$

and $\|g_n\|_{L^2}$ stays bounded, we have that $F_j(f_{jn}, g_n) \rightarrow 0$ as $n \rightarrow \infty$. As a result,

$$\begin{aligned} \Theta(r, l, m) &= \lim_{n \rightarrow \infty} E(f_{1n}, f_{2n}, g_n) \\ &\geq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} (|\partial_x f_{1n}|^2 + |\partial_x f_{2n}|^2 + |\partial_x g_n|^2) dx \geq 0 \end{aligned}$$

which contradicts Lemma 4.1 and hence, $\mu \neq 0$.

The fact that the complex-valued function ϕ_1 is of the form $\phi_1(x) = e^{i\theta_1}\tilde{\phi}_1(x)$ with $\theta_1 \in \mathbb{R}$ and $\tilde{\phi}$ real-valued nonnegative function can be easily proved by using the first equation of (1.10). (A proof of this fact is given in Theorem 2.1 of [1] for $\gamma_1 = 0$ and in Theorem 3.7 of [8] for $\gamma_1 > 0$; same proof works in the present situation.) Similarly $\phi_2(x) = e^{i\theta_2}\tilde{\phi}_2(x)$ with $\theta_2 \in \mathbb{R}$ and $\tilde{\phi}_2 \geq 0$ on \mathbb{R} . To continue the proof, we need the following result.

Claim. Suppose $\{(f_{1n}, f_{2n}, g_n)\} \subset \mathcal{H}$ be an (r, l, m) -admissible sequence satisfying the condition (4.4). If $(r, l, m) \in \mathbb{R}_+^3$, then there exists $\delta_1, \delta_2 > 0$ such that for all n large enough, one has that

$$E_j(f_{jn}) - F_j(f_{jn}, g_n) \leq -\delta_j, \quad j = 1, 2.$$

To see this, we take $j = 1$; the proof for $j = 2$ follows same argument. We argue by contradiction. After choosing an appropriate subsequence if necessary, assume that there exists a minimizing sequence $\{(f_{1n}, f_{2n}, g_n)\}$ that satisfies

$$\liminf_{n \rightarrow \infty} [E_1(f_{1n}) - F_1(f_{1n}, g_n)] \geq 0. \quad (4.27)$$

This in turn implies that

$$\begin{aligned} \Theta(r, l, m) &= \lim_{n \rightarrow \infty} E(f_{1n}, f_{2n}, g_n) \\ &\geq \liminf_{n \rightarrow \infty} [E_2(f_{2n}) + E_3(g_n) - F_2(f_{2n}, g_n)]. \end{aligned} \quad (4.28)$$

Take the functions ϕ_l and w_m as defined in Lemma 4.3. Then, in view of (4.28), we have that $\Theta(r, l, m) \geq E_{23}(\phi_l, w_m)$. On the other hand, as in the proof of part (ii) of Lemma 4.2, take any $R_r \in S_r$ satisfying

$$\|\partial_x R_r\|_{L^2}^2 - F_1(R_r, w_m) < 0.$$

Using this inequality, it is deduced that

$$\begin{aligned} \Theta(r, l, m) &\leq E(R_r, \phi_l, w_m) \leq E_{23}(\phi_l, w_m) + (\|\partial_x R_r\|_{L^2}^2 - F_1(R_r, w_m)) \\ &< E_{23}(\phi_l, w_m), \end{aligned}$$

a contradiction. This completes the proof the claim.

Next, multiply the first and second equations of (1.9) by $\overline{\phi_1}$ and $\overline{\phi_2}$, respectively, and integrate over the real line. After suitable integrations by parts, it follows immediately from the above claim that $\sigma_1 > 0$ and $\sigma_2 > 0$. To prove the remaining assertions of Theorem 2.5, we borrow an argument from [2]. Since $\sigma_1 > 0$ and $\sigma_2 > 0$, the first two equations in (1.9) can be rewritten in the following convolution form

$$\phi_j = P_{\sigma_j} \star (\gamma_j |\phi_j|^{q_j} \phi_j + \alpha_j \phi_j w), \quad j = 1, 2, \quad (4.29)$$

where for any $a > 0$, the kernel P_a is defined via $\widehat{P}_a(k) = (s + k^2)^{-1}$. Next using the fact that

$$E(|\phi_1|, |\phi_2|, |w|) = E(|\phi_1|, |\phi_2|, w) = \Theta(r, l, m),$$

one can show that

$$\int_{-\infty}^{\infty} |\phi_j|^2 |w| dx = \int_{-\infty}^{\infty} |\phi_j|^2 w dx, \quad j = 1, 2, \quad (4.30)$$

(for details, readers may consult [2]). Then, the identity (4.30) implies that $w(x) \geq 0$ at all $x \in \mathbb{R}$ for which $\tilde{\phi}_1(x) \neq 0$ and $\tilde{\phi}_2(x) \neq 0$. It then follows from the convolution identity (4.29) that $\tilde{\phi}_1(x) > 0$ and $\tilde{\phi}_2(x) > 0$ for all $x \in \mathbb{R}$. The proof that $w(x) > 0$ goes through unchanged as in the proof of Theorem 1.1 (iv) of [2] and so will not be repeated here. \square

5. Stability Analysis for Solitary Waves

In this section, consideration is given to the full variational problem (2.14). To prove the existence of solutions to the problem (2.14), we establish a relation between the solutions to (4.1) and (2.14), following the arguments of [1, 2]. Throughout this section, we assume that all conditions of (2.11) hold and that $1 \leq p < 4/3$.

5.1. The full variational problem. We begin by showing that every minimizing sequence for $\Lambda(r, l, m)$ is bounded. By a minimizing sequence for the problem (2.14) we mean a sequence $\{(h_{1n}, h_{2n}, g_n)\} \subset \mathcal{H}$ satisfying the conditions

$$\lim_{n \rightarrow \infty} Q(h_{1n}) = r, \quad \lim_{n \rightarrow \infty} Q(h_{2n}) = l, \quad \lim_{n \rightarrow \infty} H(h_{1n}, h_{2n}, g_n) = m,$$

and

$$\lim_{n \rightarrow \infty} E(h_{1n}, h_{2n}, g_n) = \Lambda(r, l, m).$$

Lemma 5.1. *If $\{(h_{1n}, h_{2n}, g_n)\}$ is a minimizing sequence for (2.14), then there exists a constant $B > 0$ such that*

$$\|h_{1n}\|_{H^1} + \|h_{2n}\|_{H^1} + \|g_n\|_{H^1} \leq B, \quad \text{for all } n.$$

Proof. We begin by estimating the sum of the component masses. Because $Q(h_{jn}), j = 1, 2$, stay bounded, it then follows that

$$\begin{aligned} J_2 &\equiv \|g_n\|_{L^2}^2 + \sum_{j=1}^2 \|h_{jn}\|_{L^2}^2 = \left| H(\Delta_n) - 2\text{Im} \int_{-\infty}^{\infty} \sum_{j=1}^2 h_{jn} \overline{\partial_x h_{jn}} dx \right| + \sum_{j=1}^2 \|h_{jn}\|_{L^2}^2 \\ &\leq C \left(1 + \sum_{j=1}^2 \|h_{jn}\|_{L^2}^2 \right) + \sum_{j=1}^2 \|h_{jn}\|_{L^2}^2 \leq C (1 + \|\Delta_n\|_{\mathcal{H}}), \end{aligned} \quad (5.1)$$

where $\Delta_n = (h_{1n}, h_{2n}, g_n)$ and $C = C(r, l, m)$. Define the quantity $\mathcal{L}_p(bx, cy) = b|x|^{p+2} + c|x|^2y$. Then it follows directly from (5.1) that

$$\begin{aligned} \|\Delta_n\|_{\mathcal{H}}^2 &= E(\Delta_n) + \int_{-\infty}^{\infty} \left(\tau g_n^{p+2} + \sum_{j=1}^2 \mathcal{L}_{q_j}(\tau_j h_{jn}, \alpha_j g_n) \right) dx + J_2 \\ &\leq C \|g_n\|_{L^{p+2}}^{p+2} + C \int_{-\infty}^{\infty} \sum_{j=1}^2 \mathcal{L}_{q_j}(h_{jn}, |g_n|) dx + C(1 + \|\Delta_n\|_{\mathcal{H}}). \end{aligned} \quad (5.2)$$

The Gagliardo-Nirenberg inequality together with the estimate of $\|g_n\|_{L^2}^2$ as in (5.1) assures that

$$\|g_n\|_{L^{p+2}}^{p+2} \leq C \left(\|\Delta_n\|_{\mathcal{H}}^{p/2} + \|\Delta_n\|_{\mathcal{H}}^{(3p+4)/4} \right). \quad (5.3)$$

Similarly, one can estimate

$$\begin{aligned} \mathcal{L}_{q_j}(h_{jn}, |g_n|) &\leq C \|\Delta_n\|_{\mathcal{H}}^{q_j/2} + C \|\partial_x h_{jn}\|_{L^2}^{1/2} \|g_n\|_{L^2} \\ &\leq C \left(1 + \|\Delta_n\|_{\mathcal{H}} + \|\Delta_n\|_{\mathcal{H}}^{q_j/2} \right), \quad j = 1, 2. \end{aligned} \quad (5.4)$$

Applying the estimates (5.3) and (5.4), it follows from (5.2) that

$$\|\Delta_n\|_{\mathcal{H}}^2 \leq C \left(1 + \|\Delta_n\|_{\mathcal{H}} + \|\Delta_n\|_{\mathcal{H}}^{p/2} + \|\Delta_n\|_{\mathcal{H}}^{(3p+4)/4} + \sum_{j=1}^2 \|\Delta_n\|_{\mathcal{H}}^{q_j/2} \right),$$

which in turn implies that $\|\Delta_n\|_{\mathcal{H}}$ is bounded. \square

In the following lemma we relate the solutions of (4.1) to that of (2.14).

Lemma 5.2. *Suppose that $(r, l, m) \in \mathbb{R}_+^2 \times \mathbb{R}$, and let $b = b_{r,l,m}(A)$ be as defined in (2.15). Then the following holds*

$$\Lambda(r, l, m) = \inf \{ \Theta(r, l, A) + b^2(r + l) : A \geq 0 \}. \quad (5.5)$$

Furthermore, if $\{(h_{1n}, h_{2n}, g_n)\} \subset \mathcal{H}$ is a minimizing sequence for the problem $\Lambda(r, l, m)$, then there exist a subsequence $\{(h_{1n_k}, h_{2n_k}, g_{n_k})\}$ and a number $A \geq 0$ such that the sequence

$$\{ (e^{ib_{r,l,m}(A)x} h_{1n_k}, e^{ib_{r,l,m}(A)x} h_{2n_k}, g_{n_k}) \}$$

of functions in \mathcal{H} forms a minimizing sequence for $\Theta(r, l, A)$. Moreover, we have that

$$\Lambda(r, l, m) = \Theta(r, l, A) + b(r + l). \quad (5.6)$$

Furthermore, one has $A > 0$ provided that $\gamma_1 = \gamma_2 = 0$.

Proof. To prove (5.5), suppose first that $A \geq 0$ and let $(h_1, h_2, g) \in S_{r \times l} \times K_A$ be given. Let $b = b_{r,l,m}(A)$ be as defined in (2.15) and

$$c_j = \operatorname{Im} \int_{-\infty}^{\infty} h_j \overline{\partial_x h_j} dx, \quad j = 1, 2.$$

Put $f_j(x) = e^{ik_j x} h_j(x)$ with $k_1 = (c_1/r) - b$ and $k_2 = (c_2/l) - b$. Then, for $\Delta = (f_1, f_2, g)$ and $U = (h_1, h_2, g)$, an elementary calculation gives

$$H(\Delta) = H(U) - 2 \sum_{j=1}^2 k_j \|h_j\|_{L^2}^2 = A + 2(c_1 + c_2) - 2(k_1 r + k_2 l) = m.$$

Since $Q(f_1) = Q(h_1) = r$ and $Q(f_2) = Q(h_2) = l$, we conclude that

$$\begin{aligned} \Lambda(r, l, m) &\leq E(\Delta) = E(U) + \sum_{j=1}^2 k_j^2 \|h_j\|_{L^2}^2 - 2 \sum_{j=1}^2 k_j \operatorname{Im} \int_{-\infty}^{\infty} h_j \overline{\partial_x h_j} dx \\ &= E(U) + b^2(r + l) - \frac{c_1}{r} - \frac{c_2}{l} \leq E(U) + b^2(r + l). \end{aligned} \quad (5.7)$$

One can now take infimum over the set $S_{r \times l} \times K_A$ to obtain

$$\Lambda(r, l, t) \leq \inf \{ \Theta(r, l, A) + b^2(r + l) : A \geq 0 \}. \quad (5.8)$$

To obtain the reverse inequality, let $(r, l, m) \in \mathbb{R}_+^2 \times \mathbb{R}$ be given and $U = (h_1, h_2, g) \in \mathcal{H}$ be such that $(h_1, h_2) \in S_{r \times l}$ and $H(U) = m$. We will show that there exists $A \geq 0$ such that

$$E(U) \geq \Theta(r, l, A) + b^2(r + l).$$

Choose $A = \|g\|_{L^2}^2$. Then, by the definition of H , we have that

$$A = m - 2 \sum_{j=1}^2 \operatorname{Im} \int_{-\infty}^{\infty} h_j \overline{\partial_x h_j} dx.$$

For $j = 1, 2$, define $f_j(x) = e^{ib_{r,l,m}(A)x} h_j(x)$, where $b = b_{r,l,m}(A)$ is as defined in (2.15). Then, a straightforward calculation yields

$$\begin{aligned} E(\Delta) &= E(U) + \sum_{j=1}^2 b^2 \|h_j\|_{L^2}^2 - 2 \sum_{j=1}^2 b \operatorname{Im} \int_{-\infty}^{\infty} h_j \overline{\partial_x h_j} dx \\ &= E(U) + b^2(r + l) - b(m - A) = E(U) - b^2(r + l), \end{aligned}$$

from which it is obvious that $E(U) = E(\Delta) + b^2(r + l)$. Since $Q(f_1) = Q(h_1) = r$ and $Q(f_2) = Q(h_2) = l$, and $g \in K_A$, we have that $A \geq 0$ and $E(\Delta) \geq \Theta(r, l, A)$. In consequence, one has that

$$E(U) \geq \Theta(r, l, A) + b^2(r + l) \geq \inf_{A \geq 0} \{ \Theta(r, l, A) + b^2(r + l) \}.$$

Upon taking infimum over all functions $U \in \mathcal{H}$ such that $(h_1, h_2) \in S_{r \times l}$ and $H(U) = m$, we obtain the reverse inequality

$$\Lambda(r, l, m) \geq \inf_{A \geq 0} \{ \Theta(r, l, A) + b^2(r + l) \}. \quad (5.9)$$

Putting the inequalities (5.8) and (5.9) together, we see that identity (5.5) holds.

Next, denote $\Delta_n = (h_{1n}, h_{2n}, g_n)$. The sequence $\{A_n\}$ of real numbers given by

$$A_n = \|g_n\|_{L^2}^2 = m - 2 \sum_{j=1}^2 \operatorname{Im} \int_{-\infty}^{\infty} h_{jn} \overline{\partial_x h_{jn}} dx$$

is bounded. Therefore, by extracting an appropriate subsequence, one may assume that A_n converges to $A \geq 0$. So by restricting consideration to the corresponding subsequence, let $b = b_{r,l,m}(A)$ and define $f_{jn}(x) = e^{ibx} h_{jn}(x)$. Denote $U_n = (f_{1n}, f_{2n}, g_n)$. Then one can invoke (5.5) to obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} E(U_n) &= \lim_{n \rightarrow \infty} [E(\Delta_n) + b^2(r+l) - b(m - A_n)] \\ &= \Lambda(r, l, m) - b^2(r+l) \leq \Theta(r, l, m). \end{aligned}$$

To obtain the reverse inequality, suppose first that $A > 0$. Then the sequences of numbers $\alpha_{1n} = \sqrt{r}/\|f_{1n}\|_{L^2}^2$, $\alpha_{2n} = \sqrt{l}/\|f_{2n}\|_{L^2}^2$, and $\beta_n = \sqrt{A}/\|g_n\|_{L^2}^2$ are well-defined for sufficiently large n . Since $\|\alpha_{1n}f_{1n}\|_{L^2}^2 = r$, $\|\alpha_{2n}f_{2n}\|_{L^2}^2 = l$, and $\|\beta_n g_n\|_{L^2}^2 = A$, it follows immediately that

$$\lim_{n \rightarrow \infty} E(f_{1n}, f_{2n}, g_n) = \lim_{n \rightarrow \infty} E(\alpha_{1n}f_{1n}, \alpha_{2n}f_{2n}, \beta_n g_n) \geq \Theta(r, l, m).$$

If $A = 0$, then one has that

$$\lim_{n \rightarrow \infty} E(U_n) = \lim_{n \rightarrow \infty} \sum_{j=1}^2 E_j(f_{jn}) = \Theta(r, l, 0).$$

It now follows that the relation (5.6) holds and that $E(U_n) \rightarrow \Theta(r, l, m)$, this means that $\{U_n\}$ is a minimizing sequence for $\Theta(r, l, m)$.

Finally, consider the case when $\gamma_1 = \gamma_2 = 0$. Suppose, for the sake of contradiction, that $A = 0$. Then

$$\Theta(\lambda_1, \lambda_2, 0) = \inf \left\{ \int_{-\infty}^{\infty} (|\partial_x f|^2 + |\partial_x g|^2) dx : \|f\|_{L^2}^2 = \lambda_1, \|g\|_{L^2}^2 = \lambda_2 \right\}.$$

It is clear that $\Theta(r, l, 0) \geq 0$ and an application of (5.6) gives $\Lambda(r, l, m) \geq 0$. On the other hand, let Δ_θ be as considered in Lemma 4.1. Then one obtains $E(\Delta_\theta) < 0$, which in turn implies that $\Lambda(r, l, m) < 0$, a contradiction. \square

5.2. Stability result for (2+1)-component NLS-gKdV. We now prove Theorems 2.6 and 2.8.

Proof of Theorem 2.6 (existence result). To prove part (i), using the same notation as in Lemma 5.2, we may assume by passing to an appropriate subsequence that $\{(e^{ibx} h_{1n}, e^{ibx} h_{2n}, g_n)\}$ is a minimizing sequence for $\Theta(r, l, A)$, for $A \geq 0$, $b = b_{r,l,m}(A)$, and the relation (5.6) holds. If $A > 0$, then Theorem 2.5 allows us to conclude, again possibly for a subsequence only, that there exists a family $(y_n) \subset \mathbb{R}$ such that

$$\{(e^{ib(\cdot+y_n)} h_{1n}(\cdot + y_n), e^{ib(\cdot+y_n)} h_{2n}(\cdot + y_n), g_n(\cdot + y_n))\}$$

converges in \mathcal{H} to some $U = (\phi_1, \phi_2, w)$. The same conclusion holds in the case when $A = 0$ as well (This can be easily checked using the identity obtained in the last paragraph of the proof of Lemma 5.2.) Furthermore, U is a minimizing function for $\Theta(r, l, A)$. For $j = 1, 2$, by passing to an appropriate subsequence yet again, one may assume that $e^{iby_n} \rightarrow e^{i\theta}$ for some number $\theta \in [0, 2\pi)$. It then follows immediately that

$$(h_{1n}(\cdot + y_n), h_{2n}(\cdot + y_n), g_n(\cdot + y_n)) \rightarrow (\Phi_1, \Phi_2, w),$$

in \mathcal{H} , where Φ_j are given by $\Phi_j(x) = e^{-i(bx+\theta)}\phi_j(x)$. Let us denote $V = (\Phi_1, \Phi_2, w)$. Then, a calculation similar to that made in (5.7) yields

$$\begin{aligned}\Theta(r, l, m) &= E(U) = E(V) + b^2 \sum_{j=1}^2 \|\Phi_j\|_{L^2}^2 - 2b \sum_{j=1}^2 \operatorname{Im} \int_{-\infty}^{\infty} \Phi_j \overline{\partial_x \Phi_j} \, dx \\ &= E(V) + b^2(r + l) - b(H(V) - \|w\|_{L^2}^2) = E(V) - b^2(r + l).\end{aligned}$$

Then, from the relation (5.6), it follows that V is a minimizing function for the problem $\Lambda(r, l, m)$, which completes the proof.

To prove part (ii), let (Φ_1, Φ_2, w) be a solution of (2.14). By the first part of Lemma 5.2, it follows that $(e^{ibx}\Phi_1, e^{ibx}\Phi_2, w)$ is a minimizing sequence (and hence a minimizer) for $\Theta(r, l, \|w\|_{L^2}^2)$, where b is as defined in (2.15) with $A = \|w\|_{L^2}^2$. Then, invoking Theorem 2.5, there exists $(\theta_j, \phi_j) \in \mathbb{R} \times H_+^1(\mathbb{R})$ such that

$$(e^{ibx}\Phi_1, e^{ibx}\Phi_2) = (e^{i\theta_1}\phi_1, e^{i\theta_2}\phi_2).$$

Furthermore, if $\gamma_1 = \gamma_2 = 0$, then the last assertion of Lemma 5.2 implies that $A > 0$. Since (ϕ_1, ϕ_2, w) belongs to $\mathcal{O}_{r,l,A}$, Theorem 2.5 guarantees that $w(x) > 0$ for $x \in \mathbb{R}$. \square

Finally, we prove the stability result.

Proof of Theorem 2.8 (stability result). Part (i) is an easy consequence of the existence result (Theorem 2.6). To prove part (ii), suppose that $\mathcal{P}_{r,l,m}$ is not stable. Then there exists a sequence of solutions $\{(u_{1n}, u_{2n}, v_n)\}$ of (1.1) and a sequence of times $\{t_n\}$ such that $(u_{1n}(\cdot, 0), u_{2n}(\cdot, 0), v_n(\cdot, 0))$ converges to $\mathcal{P}_{r,l,m}$, but $(u_{1n}(\cdot, t_n), u_{2n}(\cdot, t_n), v_n(\cdot, t_n))$ does not converge to $\mathcal{P}_{r,l,m}$ in \mathcal{H} . Since E, Q , and H are constants of the motion of (1.1) and are continuous on X , it follows that

$$\begin{aligned}\lim_{n \rightarrow \infty} Q(u_{1n}(\cdot, t_n)) &= r, \quad \lim_{n \rightarrow \infty} Q(u_{2n}(\cdot, t_n)) = l, \\ \lim_{n \rightarrow \infty} H(u_{1n}(\cdot, t_n), u_{2n}(\cdot, t_n), v_n(\cdot, t_n)) &= m, \text{ and} \\ \lim_{n \rightarrow \infty} E(u_{1n}(\cdot, t_n), u_{2n}(\cdot, t_n), v_n(\cdot, t_n)) &= \Lambda(r, l, m).\end{aligned}$$

Hence, from part (i), it follows that $(u_{1n}(\cdot, t_n), u_{2n}(\cdot, t_n), v_n(\cdot, t_n))$ converges to $\mathcal{P}_{r,l,m}$ in \mathcal{H} , which is a contradiction.

To prove part (iii), suppose $(\Phi_1, \Phi_2, w_1) \in \mathcal{P}_{r_1, l_1, m_1}$ and $(\Phi_3, \Phi_4, w_2) \in \mathcal{P}_{r_2, l_2, m_2}$, where $(r_1, l_1, m_1) \neq (r_2, l_2, m_2)$. We wish to prove that $(\Phi_1, \Phi_2, w_1) \neq (\Phi_3, \Phi_4, w_2)$. If $r_1 \neq r_2$, then the desired conclusion is clear. So assume that $r_1 = r_2$ and $m_1 \neq m_2$. Let us denote

$$\eta_1 = \frac{\|w_1\|_{L^2}^2 - m}{2(r_1 + l_1)} \text{ and } \eta_3 = \frac{\|w_2\|_{L^2}^2 - m}{2(r_2 + l_2)}.$$

Then, part (ii) of Theorem 2.6, there exists a pair of real numbers θ_1, θ_3 and a pair of \mathbb{R} -valued functions ϕ_1, ϕ_3 such that

$$\Phi_1(x) = e^{i(\eta_1 x + \theta_1)} \phi_1(x) \text{ and } \Phi_3(x) = e^{i(\eta_3 x + \theta_3)} \phi_3(x). \quad (5.10)$$

One may assume that $\Phi_1 = \Phi_3$, since otherwise the desired conclusion follows. Then (5.10) implies that

$$e^{i((\eta_1 - \eta_3)x + (\theta_1 - \theta_3))} = \phi_3(x) / \phi_1(x)$$

is a \mathbb{R} -valued function on \mathbb{R} , and hence we must have $\eta_1 = \eta_3$. Since $r_1 = r_2$, this in turn gives

$$l_2 (\|w_1\|_{L^2}^2 - m_1) = l_1 (\|w_2\|_{L^2}^2 - m_2). \quad (5.11)$$

If $l_1 \neq l_2$, then $\|\Phi_2\|_{L^2}^2 \neq \|\Phi_4\|_{L^2}^2$, and hence $\Phi_2 \neq \Phi_4$, the conclusion follows. So we may assume that $l_1 = l_2$. Then (5.11) implies $\|w_1\|_{L^2}^2 - m_1 = \|w_2\|_{L^2}^2 - m_2$ and since $t_1 \neq m_2$, this in turn implies $\|w_1\|_{L^2}^2 \neq \|w_2\|_{L^2}^2$, and hence $w_1 \neq w_2$. This completes the proof of Theorem 2.8.

Acknowledgment. The authors are thankful to Professor John Albert and Professor Felipe Linares for their teaching. A. J. Corcho was partially supported by CAPES and CNPq/Edital Universal - 481715/2012-6, Brazil and M. Panthee acknowledges supports from Brazilian agencies FAPESP 2012/20966-4 and CNPq 479558/2013-2 (Edital Universal) & 305483/2014-5.

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